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FORCED VIBRATIONS OF A MASS ON A NONLINEAR SPRING

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Pak, Chol Hui

FORCED VIBRATIONS OF A MASS ON A NONLINEAR SPRING

by

Pak, Chol Hui

Lieutenant (jg), Republic of Korea Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

IN

MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1961

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FORCED VIBRATIONS OF A MASS ON A NONLINEAR SPRING

by

Pak, Chol Hui

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING
from the
United States Naval Postgraduate School

ABSTRACT

In a recent paper* by T. J. Harvey, there is presented the idea of employing known solutions of _free_ vibrations of nonlinear single-degree conservative systems to obtain solutions of _forced_ vibrations of these systems. The forcing function is taken to be of the same form as the free response and the solutions are obtained quite easily. Harvey makes the point that while this excitation is not the sinusoidal excitation that many analysts have attempted to study, for many purposes there is reason to prefer some alternate form of excitation if by doing so useful solutions are readily obtained. In a paper of several years back, M. A. Sadowsky* derived the characteristics of several nonlinear spring systems for which the free vibrations could be obtained in rather simple closed forms. In particular, Sadowsky introduced two new springs which give solutions in terms of the elementary functions and several others which give solutions in terms of the Jacobi elliptic functions.

Since there are actually only a few fully developed cases of forced nonlinear vibrations available to analysts for testing the power and accuracy of analytical methods and for other purposes, it appeared to be of interest to apply the viewpoint of Harvey to several of the particular cases given by Sadowsky so as to work out all the details of the corresponding cases of forced vibration and thus afford several new examples. The major content of this thesis is the full development of these cases.

In addition, however, the viewpoint of Harvey is applied to two cases of piecewise linear springs simply for the purpose of illustrating such

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*See references on page 41.

applications and for the purpose, as well, of obtaining additional cases of nonlinear vibrations.

Also, since the forcing functions differ from the sinusoidal functions that many analysts have used, it was considered desirable to obtain and exhibit a simple measure of the degree of "distortion" of the actual forcing function as compared to the sinusoidal shape. Likewise, the displacements are not sinusoidal functions of time, and the "distortion" of the response (as compared to a sine curve) is also examined.

This work was done largely during the spring of 1961 at the U. S. Naval Postgraduate School, Monterey, California. The writer wishes to express his appreciation to Professor J. E. Brock for having suggested the topic and for guidance during its development.

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SYMBOLS

t = time = independent variable, chosen so that $y(0) = 0$

θ, τ = dimensionless independent variables

$y(t)$ = deflection, normalized

$F(t)$ = external force applied to the system

$S(y)$ = spring function such that $S(0) = 0$ and $S'(0) = 1.0$
$S(y)$ has the same dimension as does y

ρ = assignable dimensionless constant as defined in the relation (5)

m = mass or moment of inertia of the system

k = spring constant for a small deflection

$\omega_n = \sqrt{k/m}$ = natural frequency for a small free oscillation

$T_n = \frac{2\pi}{\omega_n}$ = period of a small free oscillation

T_0 = period of a free vibration ($\rho = 0$)

ω = frequency of vibrations

$T = \frac{4K}{\omega}$ = period of forced vibrations

$\bar{y} = y(T/4)$ = amplitude of deflection

$-$ denotes the evaluation at $t = T/4$

α = modulus of Jacobi elliptic function when a solution contains such functions (refer to Table 1)

α_1 = modulus of Jacobi elliptic functions when a spring function contains such functions (refer to Table 1)

K = the complete elliptic integral of the first kind with modulus α

K_1 = the complete elliptic integral of the first kind with modulus α_1

A = the distortion of $y(t)$ wave form as defined in Eq. (17)

B = the distortion of $F(t)$ wave form as defined in Eq. (18)

β = a constant for the springs of cubic parabola types such as
$R(Y) = Y + \beta Y^3$

Y = deflection which is not normalized

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1. Introduction.

The equation $m\ddot{y} + ky = F_0 \sin \omega t$ (1)

describes the behavior of a linear single-degree-of-freedom conservative system subjected to a sinusoidal excitation.

A natural extension for the study of a nonlinear conservative system is

$$m\ddot{y} + kS(y) = F_0 \sin \omega t \qquad (2)$$

where $S(y)$ is a nonlinear spring function, such as $S(y) = y + y^3$. Many investigators have studied Eq. (2) in detail and obtained solutions which indicate the effects of parameter ω upon y. But Eq. (2) is complicated to solve analytically, and in most cases the solutions are obtained by approximation methods.

T. J. Harvey (ref. 1) suggested a simple analytical method of studying the forced vibrations of nonlinear systems by making use of the solutions for free vibrations, i.e., where the right hand side of Eq. (2) is zero. Some years ago M. A. Sadowsky (ref. 2) presented simple solutions of free vibrations for various nonlinear springs.

This thesis applies Harvey's method to several of the cases given by Sadowsky, thus obtaining several interesting solutions for forced vibration of nonlinear systems.

In addition, a study is done on the following:

(a) The wave form of $y(t)$ is compared with sinusoidal one which is that of a linear system.

(b) Traditionally the right hand side of Eq. (2) is a sinusoidal function, as shown. By Harvey's method it differs from a sinusoidal function, but is still a periodic function. These differences are compared.

(c) Harvey's method can be applied to other than Sadowsky's nonlinear springs. It is applied to the spring characteristics which consist of linear segments, as examples.

2. Basic equations.

The forced vibration of an undamped nonlinear single-degree-of-freedom system is described by the equation

$$m\ddot{y} + k S(y) = F(t) \tag{3}$$

where $S(y)$ is a nonlinear spring function normalized so that $S(0) = 0$ and $S'(0) = 1.0$, and $F(t)$ is the external force applied to the system.

Eq. (3) is simplified by substituting:

$$\omega_n^2 = k/m \quad \text{and} \quad \tau = \omega_n t \tag{4}$$

Then

$$\frac{d^2y}{d\tau^2} + S(y) = \frac{1}{k} F(t) \tag{3a}$$

Eq. (3a) as it stands, is of great difficulty. Even if $F(t)$ is periodic and if we presume (by way of analogy with linear systems) that slight damping present in the system but not accounted for in the mathematical description causes the "transient" to die out, the "steady state" solution is generally not obtainable except by difficult analyses leading to approximations. However, the analysis is greatly simplified by henceforth confining attention to only such $F(t)$ as satisfy the relation

$$\rho = \frac{F(t)}{k S[y(t)]} \tag{5}$$

where ρ is a dimensionless constant, either positive or negative. Such a forcing function is called a "natural forcing function" (ref. 1). A positive ρ means the external force is in-phase with restoring force, a negative ρ means out-of-phase (180°), and $\rho = 0$ means free vibration.

Substituting the relation (5) into Eq. (3a), we obtain

$$\frac{d^2y}{d\tau^2} + (1-\rho) S(y) = 0 \tag{6}$$

This is further simplified by putting

$$\theta = \sqrt{1-\rho}\ \tau \tag{7}$$

Then $$\frac{d^2y}{d\theta^2} + S(y) = 0 \tag{8}$$

which is described as the case of the pseudo-free oscillation. To solve Eq. (8), let the initial conditions be

$$y(0) = 0 , \qquad \left.\frac{dy}{d\theta}\right)_0 = V_0$$

The potential energy stored in the spring is

$$U(y) = \int_0^y S(y)\,dy \; , \qquad U(0) = 0$$

Furthermore, by conservation of energy, V_0 is related as

$$V_0^2 = 2\,U(\bar{y})$$

where $\bar{y}$ is the amplitude of deflection. If T/4 is defined as the quarter-period of oscillation, $\bar{y}$ can be written as $\bar{y} = y(T/4)$.

Eq. (8) is integrated and simplified as

$$\left(\frac{dy}{d\theta}\right)^2 = 2\,U(\bar{y}) - 2\,U(y) \tag{9}$$

Integration of Eq. (9) leads to

$$\theta = \int_0^y \frac{dy}{\left[2U(\bar{y}) - 2U(y)\right]^{1/2}} \tag{10}$$

Again Eq. (10) can be written in the following general form with $\bar{y}$ as a parameter,

$$\theta = \theta(y, \bar{y}) \tag{11}$$

The inverse form of Eq. (11) can be expressed as

$$y = y(\theta, \bar{y})$$

Since the dimensionless variable θ is closely related to time t, using Eqs. (4) and (7),

$$\theta = \omega_n t \sqrt{1-\rho} \tag{12}$$

Then $$y = y(\omega_n t\sqrt{1-\rho}\, , \bar{y}) \tag{13}$$

This is the solution of the forced vibration of Eq. (3) in which the external forces $F(t)$ satisfy the relation (5). For free vibrations $F(t) = 0$ and $\rho = 0$. Then the solutions of free vibrations are from Eq. (13),

$$y = y(\omega_n t ,\ \bar{y}) \tag{14}$$

Comparing Eqs. (13) and (14), the solutions of forced vibrations are obtained from those of free vibrations by multiplying the independent variable t(= time) by the factor $\sqrt{1-\rho}$. The assignable dimensionless constant ρ is the ratio of the external force to the restoring force, namely

$$\rho = \frac{\bar{F}}{k S(\bar{y})}$$

where the relation (5) is rewritten. The amplitude of external force $\bar{F}$ is given as 2.0 k or 3.5 k , for example, which means the force unit corresponding to the value of the spring constant (at small deflection) multiplied by 2.0 or 3.5 units of deflection respectively. Or in other terms, Eq. (3a) represents a forced vibration wherein $S(y)$ plays the role of a restoring "force" and $\frac{1}{k} F(t)$ plays the role of an exciting force; the time scale has been chosen so that both "forces" have the same dimension as the "deflection response", $y(t)$. Thus we can say that the amplitude of exciting force ($\bar{F}/k$) has magnitude 2.0 or 3.5, say. For every amplitude of deflection $\bar{y}$, the values of $\bar{S} = S(\bar{y})$ are found, which, in turn, determines the assignable constant ρ .

Sadowsky's paper gives the essential details for ten cases of nonlinear springs for which the free oscillations are relatively simple. Two of the solutions are in terms of elementary functions, and the others involve the Jacobi elliptic functions. Corresponding to each of these cases, by using Harvey's method, it is possible to construct a complete solution for the forced oscillations resulting from "natural" excitation. In this thesis four of the cases are developed in detail--the two cases involving only elementary functions and two of the cases (cubically hardening and cubically softening -- i.e., Duffing's spring) involving elliptic functions. Also, two cases of piecewise-linear springs (cases not included among those given by Sadowsky) are also developed.

Sadowsky's solutions are expressed in terms of the initial velocity V_0, which can be replaced by $\bar{y}$, for convenience, with the energy relation $V_0^2 = 2\,U(\bar{y})$ in the conservative systems. The solutions are tabulated in Table 1 and their approximate spring characteristics are shown in Fig. 1. The symbol α represents the modulus of Jacobi functions, K means the complete elliptic integral of the first kind with modulus α.

In Fig. 1, the numbers in circles represent the spring numbers as indicated in Table[1]. Springs marked as 1, 2, 3, and 4 are drawn heavily. The family springs 7 is in between springs 1 and 4 depending upon the modulus α_1, where for $\alpha_1 = 0$ it coincides with spring 1 and for $\alpha_1 = 1.0$ it coincides with spring 4. The space between springs 4 and 3 is filled with the family of springs 6 depending upon the modulus α_1. For $\alpha_1 = 0$ it coincides with spring 3, and for $\alpha_1 = 1.0$ it coincides with spring 4. Each one reaches a maximum and drops down to zero. The family of springs 5 is of softening type, and so is the elementary spring 2 obtainable with $\alpha_1 = 1.0$. Springs 8, 9, and 10 are cubic parabolas, they are not shown in Fig. 1 since they can readily be visualized.

Table 1 - The solutions of forced vibrations for Sadowsky's springs (adapted from Sadowsky; ref. 2)

The solutions of forced vibrations such as

$$m\ddot{y} + k\,S(y) = F(t)$$

are tabulated below, in which $F(t)$ satisfies the relation (5).

	$S(y)$	Solutions	Modulus α	$\frac{\omega}{\omega_n\sqrt{1-\rho}}$
1	$\sin y/\cos^3 y$	$\sin y = \sin\bar{y}\,\sin\omega t$	—	$\sec\bar{y}$
2	$\sinh y/\cosh^3 y$	$\sinh y = \sinh\bar{y}\,\sin\omega t$	—	[illegible]
3	$\frac{1}{2}\sin 2y$	$\sin y = \sin\bar{y}\,\mathrm{sn}(\omega t,\alpha)$	$\sin\bar{y}$	1
4	$\frac{1}{2}\sinh 2y$	$\tanh y = \tanh\bar{y}\,\mathrm{sn}(\omega t,\alpha)$	$\tanh\bar{y}$	$\cosh\bar{y}$
5*	$s\,d\,c$	$\mathrm{cn}(y-K_1,\alpha_1) = \mathrm{cn}(\bar{y}-K_1,\alpha_1)\,\mathrm{sn}(\omega t,\alpha)$	$\mathrm{cn}(\bar{y}-K_1,\alpha_1)$	$\mathrm{dn}(\bar{y},\alpha_1)$
6*	$s\,c/d^3$	$s = \mathrm{sn}(\bar{y},\alpha_1)\,\mathrm{sn}(\omega t,\alpha)$	$\mathrm{sn}(\bar{y},\alpha_1)$	$\frac{1}{\mathrm{dn}(\bar{y},\alpha_1)}$
7*	$s\,d/c^3$	$s = \mathrm{sn}(\bar{y},\alpha_1)\,\mathrm{sn}(\omega t,\alpha)$	$\alpha_1\,\mathrm{sn}(\bar{y},\alpha_1)$	$\frac{1}{\mathrm{cn}(\bar{y},\alpha_1)}$
8	$y-\frac{1}{2}y^3$	$y = \bar{y}\,\mathrm{sn}(\omega t,\alpha)$	$\frac{\bar{y}}{\sqrt{4-\bar{y}^2}}$	$\sqrt{1-\frac{1}{4}\bar{y}^2}$
9	$y+\frac{1}{2}y^3$	$y = \bar{y}\,\mathrm{cn}(\omega t-K,\alpha)$	$\frac{\bar{y}}{\sqrt{4+2\bar{y}^2}}$	$\sqrt{1+\frac{1}{2}\bar{y}^2}$
10	$\frac{1}{2}y^3$	$y = \bar{y}\,\mathrm{cn}(\omega t-K,\alpha)$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\bar{y}$

* In cases 5, 6, and 7 ;

$s = \mathrm{sn}(y,\alpha_1)$ $c = \mathrm{cn}(y,\alpha_1)$ $d = \mathrm{dn}(y,\alpha_1)$

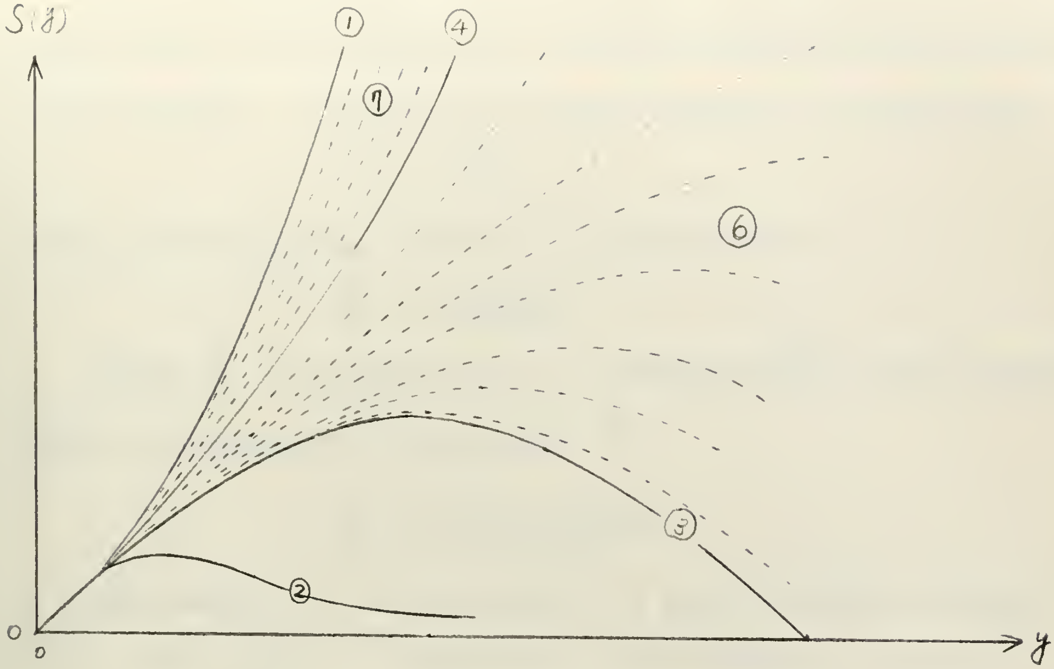

Fig. 1 Characteristics of Sadowsky's springs, (from Sadowsky, ref. 2).

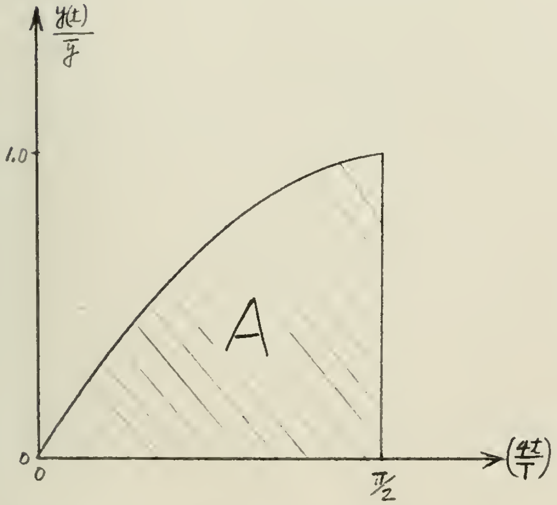

Fig. 2 Distortion of the wave form. A is defined in Eq. (17).

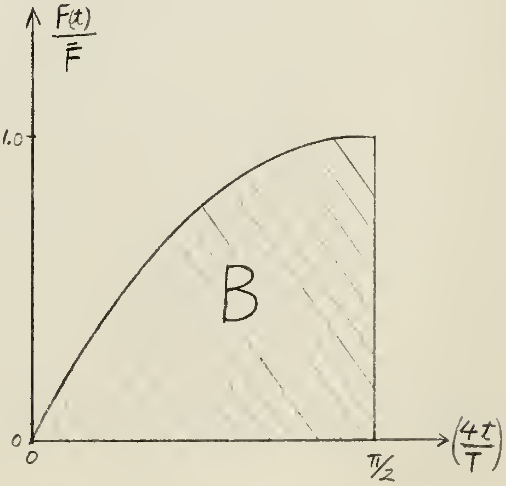

Fig. 3 Distortion of the wave form. B is defined in Eq. (18).

A. Frequency response.

When the exciting force is such that the relation (5) is satisfied, the period of forced vibration is readily obtained from the solution of free vibration. If the period of free vibration is obtained as:

$$T_0 = T_0(\bar{y})$$

then by comparing Eqs. (13) and (14), we see that

$$\frac{T}{T_0} = \frac{1}{\sqrt{1-\rho}} \tag{15}$$

Since T_0 is varying with $\bar{y}$, the ratio $\left(\frac{T}{T_0}\right)$ may be related to the period of a small free oscillation T_n.

$$\frac{T}{T_n} = \left(\frac{T_0}{T_n}\right)\frac{1}{\sqrt{1-\rho}} \tag{16}$$

This means that corresponding to a single frequency response curve of $\bar{y}$ versus $\left(\frac{T_0}{T_n}\right)$, obtained for free vibrations, we obtain a full family of response curves for forced vibrations, i.e., curves of $\bar{y}$ versus $\left(\frac{T}{T_n}\right)$ for various values of ρ.

B. Wave form of $y(t)$.

When the exciting force is such that the relation (5) is satisfied, the solution is exactly the same as that for free vibration except for the change in frequency, which is $\sqrt{1-\rho}$ times that of the free vibration. Therefore the wave form of $y(t)$ depends upon the amplitude of deflection $\bar{y}$ only, and not upon the amplitude of exciting force. When $\bar{y}$ is small, the springs are very nearly linear, and the wave forms are almost purely sinusoidal. As $\bar{y}$ increases, the wave forms of $y(t)$ are distorted from the sinusoidal forms.

Furthermore when the spring characteristics are symmetrical, the wave forms are quarter-wave symmetrical. In addition, they tend to lie between a rectangular and a triangular wave form.

To show the distortion of $y(t)$ wave forms, a measure A is defined as

$$A = \frac{\pi}{2}\left(\frac{4}{T}\right)\int_0^{T/4} \frac{y(t)}{\bar{y}}\,dt \qquad (17)$$

If we let the quarter period T/4 correspond to $\pi/2$, the measure A represents an area under the curve of $\frac{y(t)}{\bar{y}}$ as shown in Fig. 2. Note that $A=1$, for a linear spring or a small oscillation.

C. Wave forms of $F(t)$.

When the external force $F(t)$ satisfied the relation (5), the wave form of $F(t)$ is the same as that of the spring force $S[y(t)]$. Therefore it depends on $\bar{y}$ only, not the amplitude of the external force $\bar{F}$. For a symmetrical spring, the wave form of $F(t)$ is quarter-wave symmetrical in the same manner as that of $y(t)$. For a small oscillation $F(t)$ is almost a pure sinusoidal form, from which the wave form is distorted as $\bar{y}$ increases.

To compare this wave form with the traditional forcing function of sinusoidal form, a measure B is defined as

$$B = \frac{\pi}{2}\left(\frac{4}{T}\right)\int_0^{T/4} \frac{F(t)}{\bar{F}}\, dt \tag{18}$$

Note that: $B = 1.0$ for a linear spring or a small oscillation.

3. Examples.

The results of six examples are presented in Fig. (4) through Fig. (15). Figures 4, 6, 8, 10, 12, and 14 are all similar representations as follows:

(a) Ordinates are $\bar{y}$ (or $\bar{y}/c$ where c is a constant).

(b) Solid curves represent frequency response curves for labled values of $\bar{F} = F(T/4)$, including $\bar{F} = 0$, the case of free vibration.

In-phase forced vibrations are indicated by positive values of $\bar{F}$, out-of-phase by negative values. These curves relate $\bar{y}$ (ordinate) to T/T_n (bottom scale of abscissa).

(c) Distortion of $y(t)$ is indicated by a measure A, distortion of $F(t)$ by a measure B, both functions of $\bar{y}$ alone, and plotted on vertical scales.

(d) The dotted curves show spring characteristics $S(\bar{y})$ (top scale of abscissas) as functions of $\bar{y}$. In Figs. 12 and 14, this relation is shown to smaller scale in the inset and not in full scale as in the preceding figures.

Figures 5, 7, 9, 11, 13, and 15 actually display the distorted wave forms for $y(t)$ and $F(t)$.

A. Example 1 $S(y) = \frac{\sin y}{\cos^3 y}$

The characteristic of the spring is extremely hardening as shown in Fig. 4 (dotted curve). The solution of the forced vibration is already tabulated in Table 1, namely

$$\sin y = \sin \bar{y} \sin \omega t \tag{19}$$

where

$$\omega = (\omega_n \sec \bar{y}) \sqrt{1-\rho} \quad , \quad \rho = \frac{\bar{F}}{k\bar{S}} \tag{20}$$

(a) Frequency response.

From Eq. (20) we can get the solution which indicates the effects of frequency upon $\bar{y}$, making use of $T = \frac{2\pi}{\omega}$ and $T_n = \frac{2\pi}{\omega_n}$. In the dimensionless form, we get

$$\frac{T}{T_n} = \cos \bar{y} \frac{1}{\sqrt{1-\rho}} \tag{21}$$

For the given value of $\bar{F}$, ρ is obtained by dividing $\bar{F}$ by $S(\bar{y})$, then the ratio T/T_n is obtained from Eq. (21). The frequency response curves are plotted in Fig. 4.

(b) Wave form of $y(t)$.

Eq. (19) is rewritten in a convenient form,

$$\frac{y(t)}{\bar{y}} = \frac{\sin^{-1}(\sin \bar{y} \sin \omega t)}{\bar{y}} \tag{22}$$

For a small oscillation, $\lim_{\bar{y} \to 0} \frac{y(t)}{\bar{y}} = \sin \omega t$

With Eq. (22) the wave forms of $y(t)$ are plotted for $\bar{y} = 0$, 0.5, 1.0, 1.30, and $\pi/2$ in Fig. 5(a). Since Eq. (22) shows that $y(t)$ is quarter-wave symmetrical, plotting is done for a quarter period. For a small oscillation, it is clearly a pure sinusoidal form, and as $\bar{y}$ increases it is distorted. To show the distortion, the measure, A(as defined in Eq. 17) is found. The integral is not easily performed analytically so that a numerical integration method was used. The measure A is a

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function of $\bar{y}$, and is plotted along $\bar{y}$ in Fig. 4.

(c) Wave form of $F(t)$.

Using Eq. (19) and the relation (5),

$$F(t) = \rho k S[y(t)] \quad \text{and} \quad \bar{F} = \rho k S(\bar{y})$$

rearranging these into a dimensionless form, we get

$$\frac{F(t)}{\bar{F}} = \frac{\sin \omega t}{(1+\tan^2 \bar{y} \cos^2 \omega t)^{3/2}} \tag{23}$$

For a small oscillation, $\lim\limits_{\bar{y}\to 0} \dfrac{F(t)}{\bar{F}} = \sin \omega t$

which is purely sinusoidal. As $\bar{y}$ increases, it is distorted. With Eq. (23) the wave form of $F(t)$ is plotted in Fig. 5(b).

To compare $F(t)$ wave form with the traditional sinusoidal function, the measure B(as defined in Eq. 18) is obtained:

$$B = \frac{\pi}{2}\left(\frac{4}{T}\right)\int_0^{T/4} \frac{F(t)}{\bar{F}}\, dt = \cos \bar{y}$$

which is a function of $\bar{y}$ only, and is plotted in Fig. 4.

(d) Discussion.

This spring has an extremely hardening characteristic. At $\bar{y} = \frac{\pi}{2}$, the spring force becomes infinity, its slope is infinity, and the measure B is zero. For slightly smaller values of $\bar{y}$, the measure B is far away from unity. Fig. 5(b) shows that the wave form of $F(t)$ may become quite distorted.

When $\bar{y}$ is far away from the value of $\pi/2$, the measure B is not far away from unity, and Fig. 5(b) shows the wave form of $F(t)$ is not so much distorted.

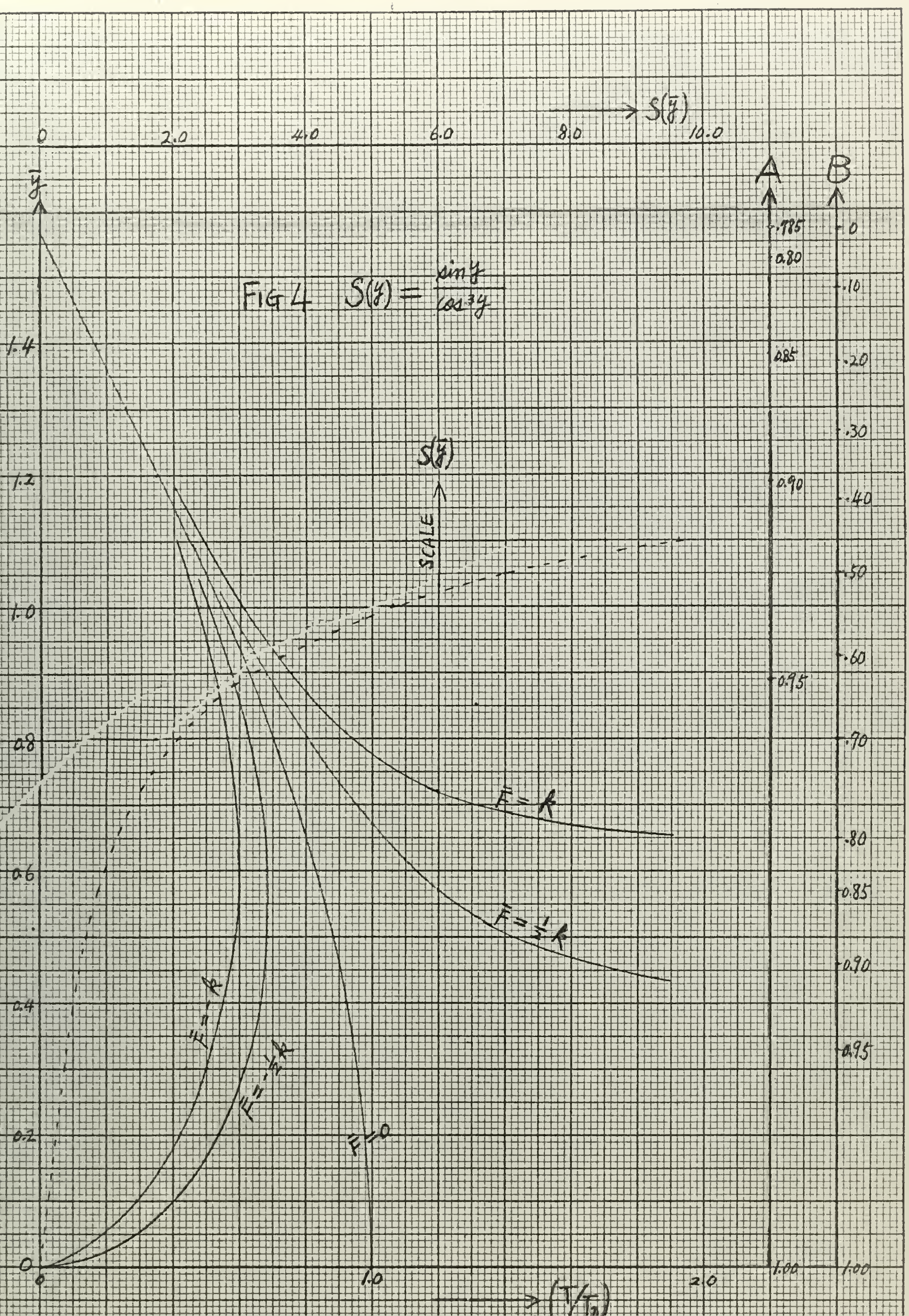


FIG 4 $S(\bar{y}) = \frac{\sin y}{\cos^3 y}$

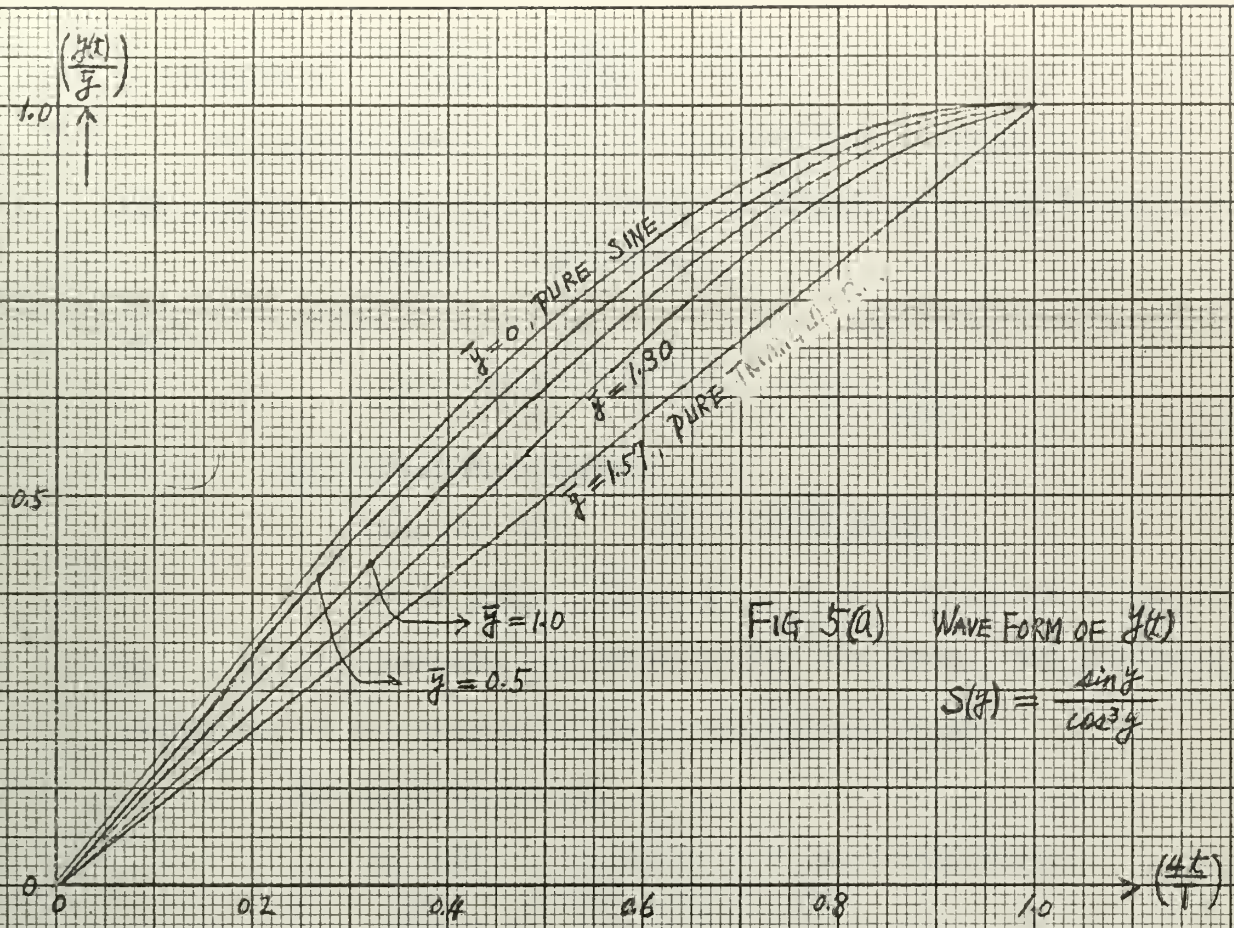


FIG 5(a) WAVE FORM OF $y(t)$

$$S(y) = \frac{\sin y}{\cos^3 y}$$

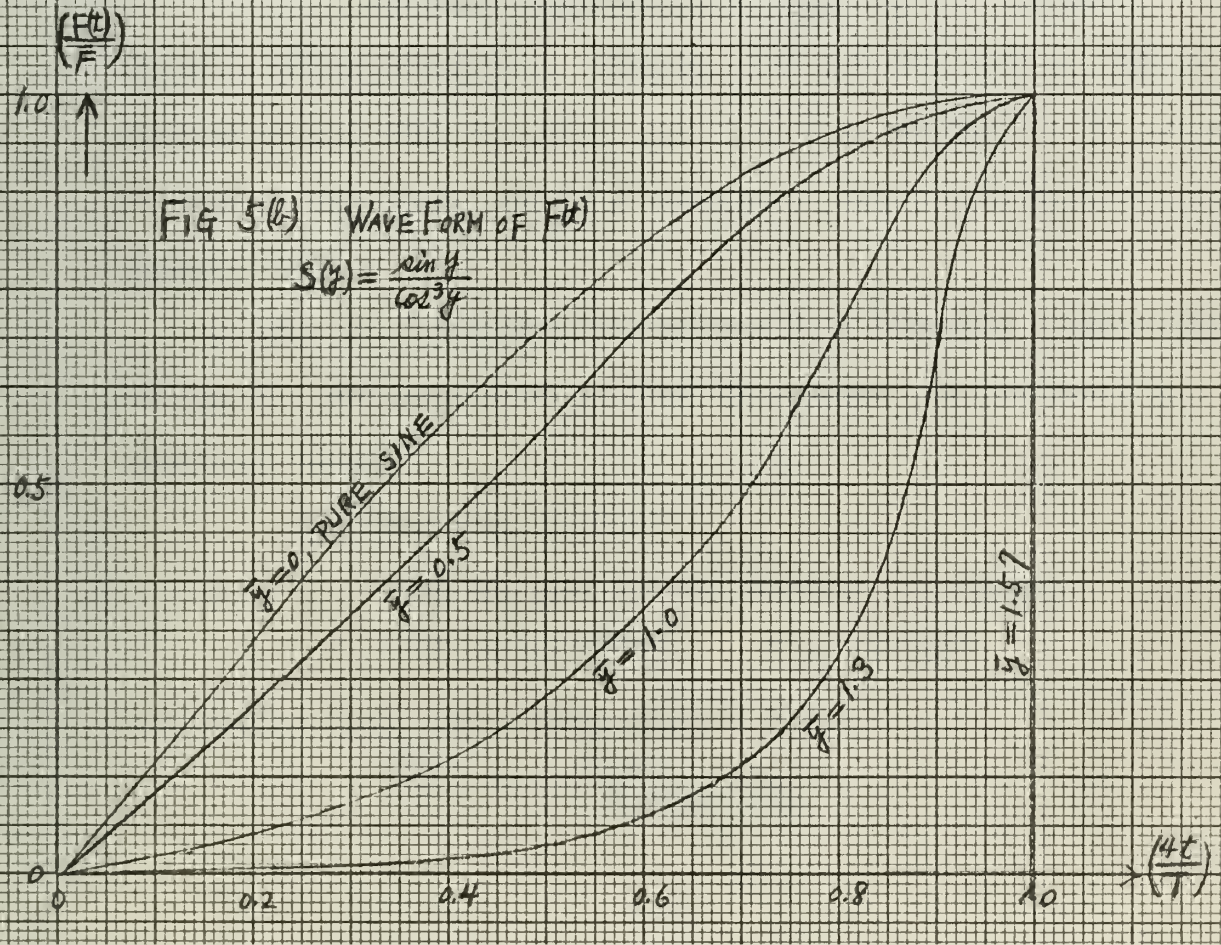


FIG 5(b) WAVE FORM OF $F(t)$

$$S(y) = \frac{\sin y}{\cos^3 y}$$

B. Example 2. $S(y) = \frac{\sinh y}{\cosh^3 y}$

This spring has a softening characteristic as shown in Fig. 6. The solution of the forced vibration is already tabulated in Table 1.

$$\sinh y = \sinh \bar{y} \sin \omega t \tag{24}$$

where

$$\omega = (\omega_n \operatorname{sech} \bar{y})\sqrt{1-\rho} \quad , \quad \rho = \frac{\bar{F}}{k\bar{S}} \tag{25}$$

(a) Frequency response.

From Eq. (25) $\frac{T}{T_n} = \cosh \bar{y} \frac{1}{\sqrt{1-\rho}}$ (26)

For a given value of $\bar{F}$, ρ is obtained as a function of $\bar{y}$, and the dimensionless values of T/T_n are found. The frequency response curves are plotted in Fig. 6 for $\bar{F}=0$ and $\pm 0.1k$.

(b) Wave for of $y(t)$.

Eq. (24) is rewritten in a convenient form,

$$\frac{y(t)}{\bar{y}} = \frac{\sinh^{-1}(\sinh \bar{y} \sin \omega t)}{\bar{y}} \tag{27}$$

For small oscillation, $\lim_{\bar{y}\to 0} \frac{y(t)}{\bar{y}} = \sin \omega t$

which is purely sinusoidal. With Eq. (27) the wave form of $y(t)$ is plotted in Fig. 7(a) for $\bar{y}=0$, 1.0, 2.0, 5.0, and ∞. As $\bar{y}$ increases it becomes distorted. And as $\bar{y}$ approaches infinity it goes to a rectangular form.

To show the distortion, the measure A is found and plotted along $\bar{y}$ in Fig. 6. The integration to find the measure A is performed by a numerical method.

(c) Wave form of $F(t)$.

Using Eq. (24) and the relation (5), we get

$$F(t) = \rho k S[y(t)] \quad , \quad \bar{F} = \rho k S(\bar{y})$$

rearranging these into a dimensionless form,

$$\frac{F(t)}{\bar{F}} = \frac{\sin \omega t}{(1 - \tanh^2 \cos^2 \omega t)^{3/2}} \tag{28}$$

For a small oscillation, $\lim_{\bar{y}\to 0} \frac{F(t)}{\bar{F}} = \sin \omega t$ which is purely sinusoidal. As $\bar{y}$ increases it is distorted. With Eq. (28) the wave form is plotted for $\bar{y} = 0, 0.5, 0.66$, and 1.0 in Fig. 7(b).

To compare $F(t)$ wave form with a sinusoidal one, the measure B is found,

$$B = \cosh \bar{y} \tag{29}$$

which is plotted along $\bar{y}$ in Fig. 6.

(d) Discussion.

The spring is of softening type: at $\bar{y} = 0.66$ $S(\bar{y})$ reaches its maximum value (which is equal to 0.384). Beyond this point it decreases and the spring force approaches zero as $\bar{y}$ approaches infinity as shown in Fig. 6. Up to $\bar{y} = 0.66$, the measure B is not far away from unity, and Fig. 7(b) shows that the wave form of $F(t)$ is not distorted appreciably.

Far beyond the point $\bar{y} = 0.66$, the value of B is much greater than unity and the wave form is greatly distorted.

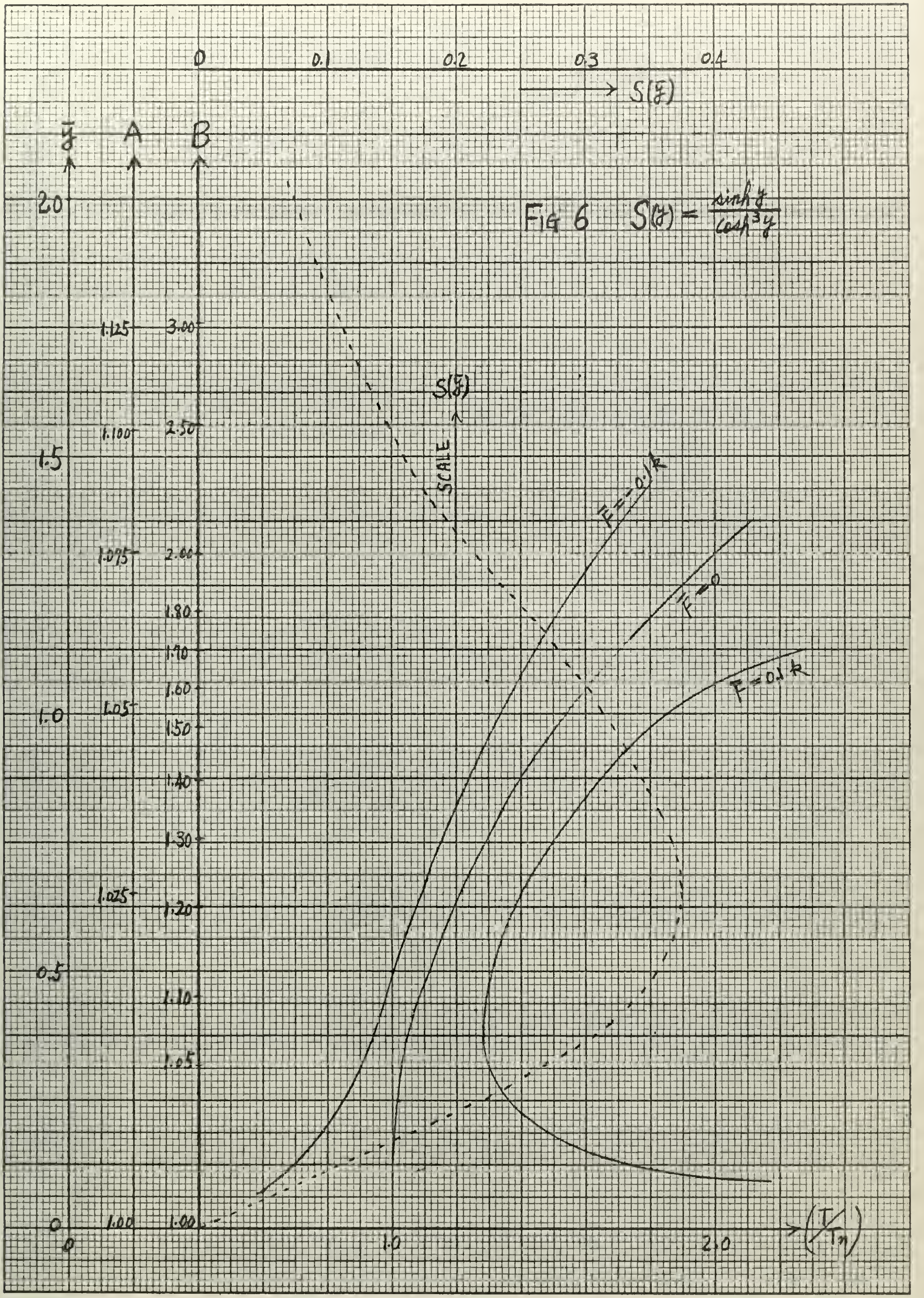


Fig 6 $S(\bar{y}) = \frac{\sinh \bar{y}}{\cosh^3 \bar{y}}$

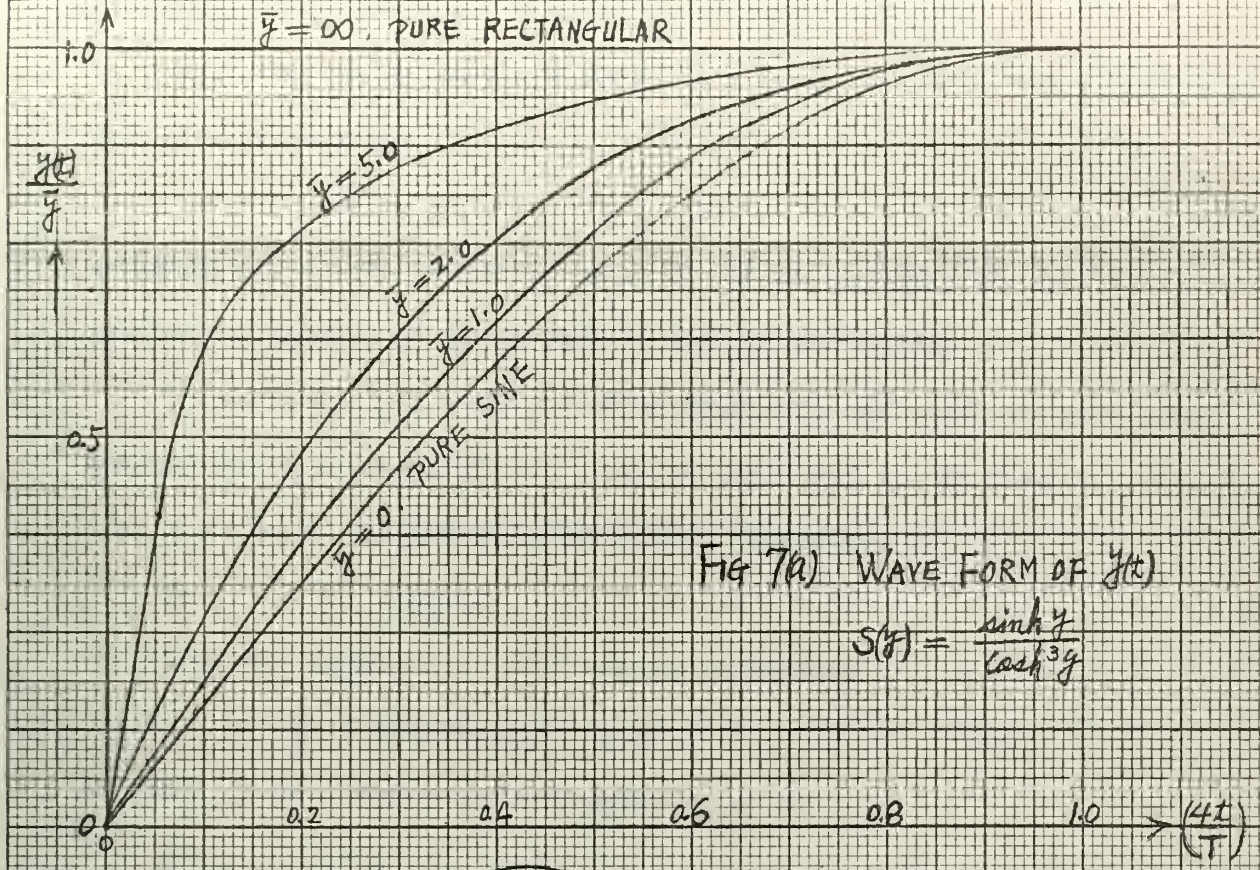


Fig 7(a) Wave Form of $y(t)$

$S(y) = \frac{\sinh y}{\cosh^3 y}$

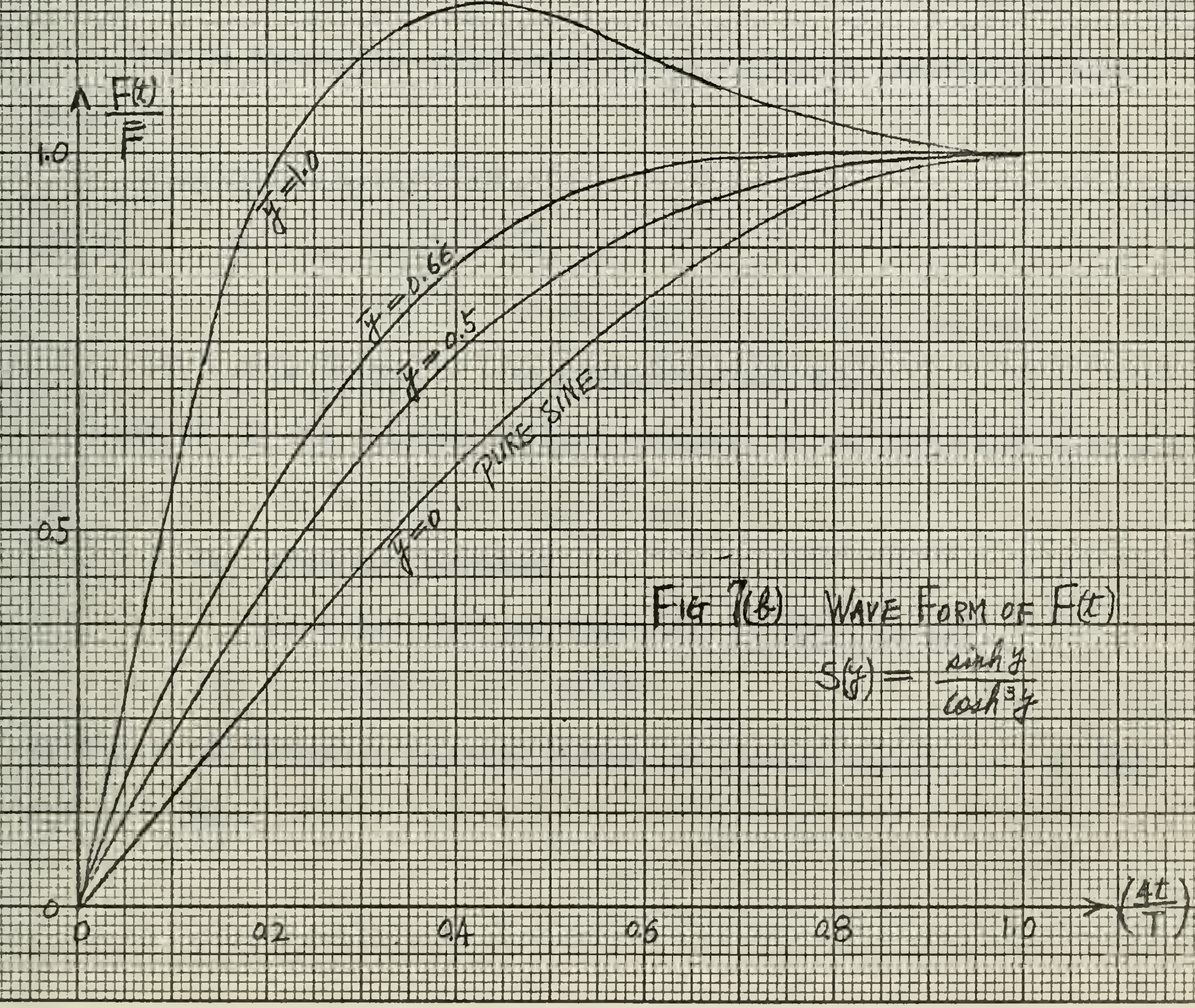


Fig 7(b) Wave Form of $F(t)$

$S(y) = \frac{\sinh y}{\cosh^3 y}$

C. Example 3. Duffing's equation. $S(y) = y + \frac{1}{2}y^3$

When the spring characteristic is a cubic parabola, the equation of forced vibration is

$$m\ddot{Y} + k(Y+\beta Y^3) = F(t) \tag{30}$$

in which Eq. (3) is rewritten here. Y is the deflection. If it is expressed in inches, β is in inch^{-2}, and k in lbs/in.

To solve Eq. (30) with the right-hand side set equal to zero, the general cubic parabola $(Y+\beta Y^3)$ can be expressed in the form of Table 1, i.e., $(y+\frac{1}{2}y^3)$ by proper substitution regardless of β.

Let $$Y = \frac{1}{\sqrt{2\beta}}\, y \tag{31}$$

Then the equation of free vibration becomes

$$\ddot{y} + \omega_n^2\left(y + \frac{1}{2}y^3\right) = 0$$

in which y is a dimensionless quantity.

Let $$S(y) = y + \frac{1}{2}y^3 \tag{32}$$

Then we obtain the solution of forced vibration immediately from Table 1.

$$y(t) = \bar{y}\, cn(\omega t - K, \alpha) \tag{33}$$

where $\omega = \omega_n\left(1+\frac{1}{2}\bar{y}^2\right)^{1/2}\sqrt{1-\rho}$

$\alpha = \bar{y}\left(1+\frac{1}{2}\bar{y}^2\right)^{-1/2}$ as the modulus of Jacobi functions (34)

$\rho = \frac{\bar{F}}{k\bar{S}}\sqrt{2\beta}$ in a dimensionless form

And K is the complete elliptic integral of the first kind with modulus α, i.e., $$K = \int_0^{\pi/2}\left(1-\alpha^2\sin^2\theta\right)^{-1/2} d\theta$$

(a) Frequency response

From Eqs. (33) and (34) with $T = \frac{4K}{\omega}$, we get

$$\frac{T}{T_n} = \frac{1}{\sqrt{1+\frac{1}{2}\bar{y}^2}}\left(\frac{2K}{\pi}\right)\frac{1}{\sqrt{1-\rho}} \tag{35}$$

In Fig. 8 the frequency response curves are plotted for $\bar{F} = 0$, $\pm\, k/\sqrt{2\beta}$, and $\pm\, 2\, k/\sqrt{2\beta}$.

(b) Wave form of $y(t)$.

Eq. (33) is rewritten:

$$\frac{y(t)}{\bar{y}} = cn(\omega t - K, \alpha)$$

For a small oscillation, $\lim_{\bar{y} \to 0} \frac{y(t)}{\bar{y}} = \sin \omega t$ which is purely sinusoidal. As $\bar{y}$ increases it becomes distorted. However there is a limit in the distortion. As $\bar{y}$ approaches infinity the value of α does not exceed $\frac{1}{\sqrt{2}}$. In Fig. 9(a) the wave form of $y(t)$ is plotted for $\bar{y} = 0$ and $\bar{y} = \infty$.

To show the distortion the measure A is found, $A = \left(\frac{\pi}{2K}\right) \frac{\cos^{-1}\sqrt{1-\alpha^2}}{\alpha}$ which is plotted in Fig. 8 along $\bar{y}$. It does not change appreciably from unity. Even if $\bar{y}$ approaches infinity, A approaches 0.942. In other words, the wave form of $y(t)$ is not distorted much from the pure sinusoidal form.

(c) Wave form of $F(t)$.

From Eq. (33) and the relation (5),

$$F(t) = \rho \frac{k}{\sqrt{2\beta}} S[y(t)] \quad , \quad \bar{F} = \rho \frac{k}{\sqrt{2\beta}} S(\bar{y})$$

rearranging these into a dimensionless form, we get

$$\frac{F(t)}{\bar{F}} = \frac{1}{1 + \frac{1}{2}\bar{y}^2} \left[cn(\omega t - K) + \frac{1}{2}\bar{y}^2 cn^3(\omega t - K) \right] \qquad (36)$$

with the modulus α.

For a small oscillation, $\lim_{\bar{y} \to 0} \frac{F(t)}{\bar{F}} = \sin \omega t$ which is purely sinusoidal. With Eq. (36) the wave form of $F(t)$ is plotted for $\bar{y} = 0$, 1.0, 2.0, and infinity in Fig. 9(b).

To compare $F(t)$ with a sinusoidal force, the measure B is found,

$$B = \frac{\pi}{2K} \left(\frac{\bar{y}^2 + 4}{2\bar{y}^2 + 4} \right)^{1/2}$$

which is plotted along $\bar{y}$ in Fig. 8.

(d) Discussion.

This spring is a hardening type as shown in Fig. 8 (dotted curve). Every type of cubic parabola, such as $Y + \beta Y^3$, can be reduced to

the form $S(Y) = Y + \frac{1}{2} Y^3$ by letting $Y = \frac{1}{\sqrt{2\beta}} y$ without losing the generality of the governing equation (Eq. 3).

Fig. 9(b) shows that the wave form of $F(t)$ is not distorted to an appreciable amount, and the measure B is not far away from unity.

The application of Harvey's method to Duffing's equation was considered in a recent note (ref. 3) which appeared after this thesis was nearly completed. For the hardening spring, considered in this section, the results given in the note agree with those given here. However, in the case of the softening spring, treated in the next section following, the results do not agree since the note contains an error. In place of the cn function in the note, the cd function should appear for the softening case, and the argument of the function should also be slightly modified.

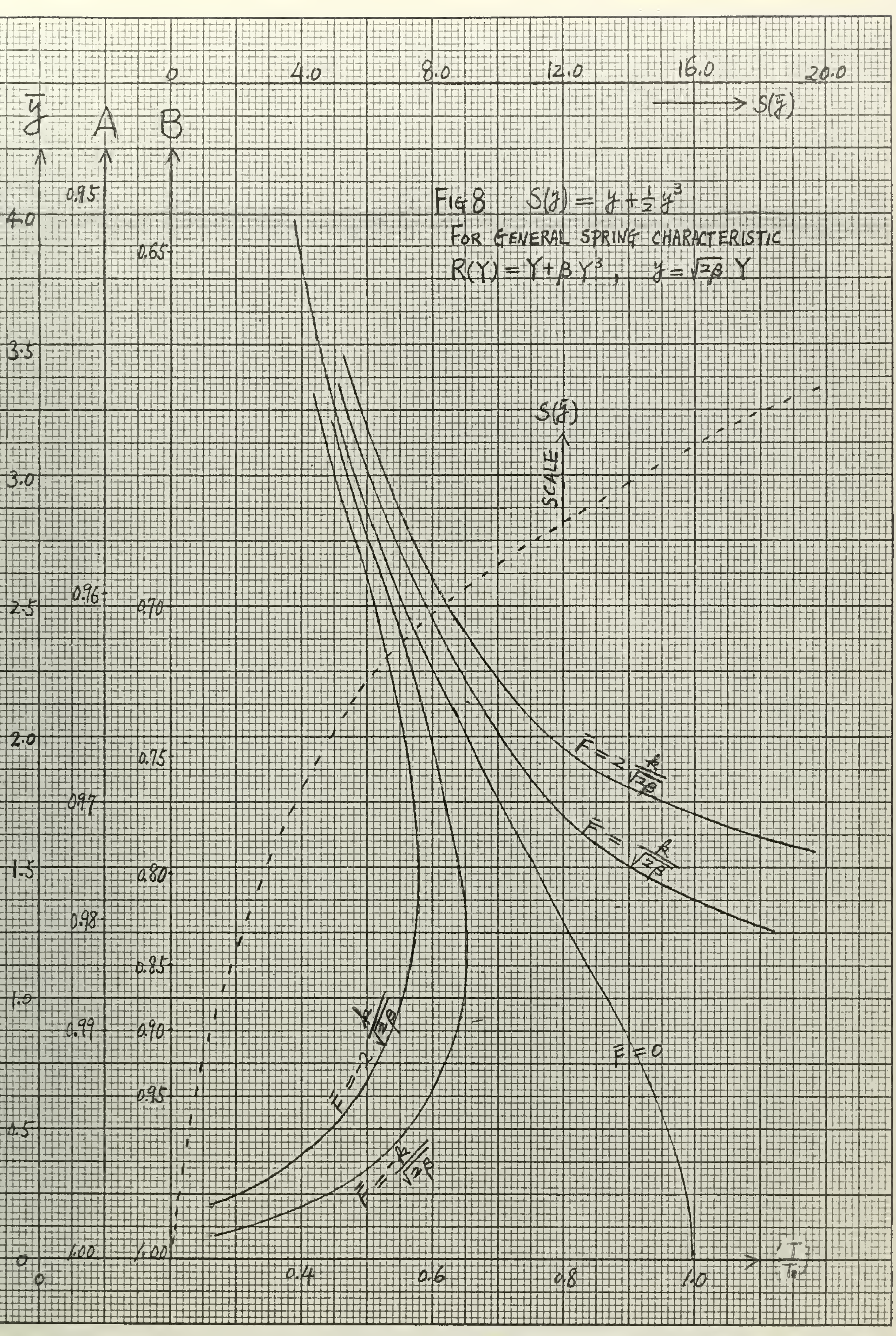


FIG 8 $S(y) = y + \frac{1}{2}y^3$

FOR GENERAL SPRING CHARACTERISTIC

$R(Y) = Y + \beta Y^3, \quad y = \sqrt{2\beta}\, Y$

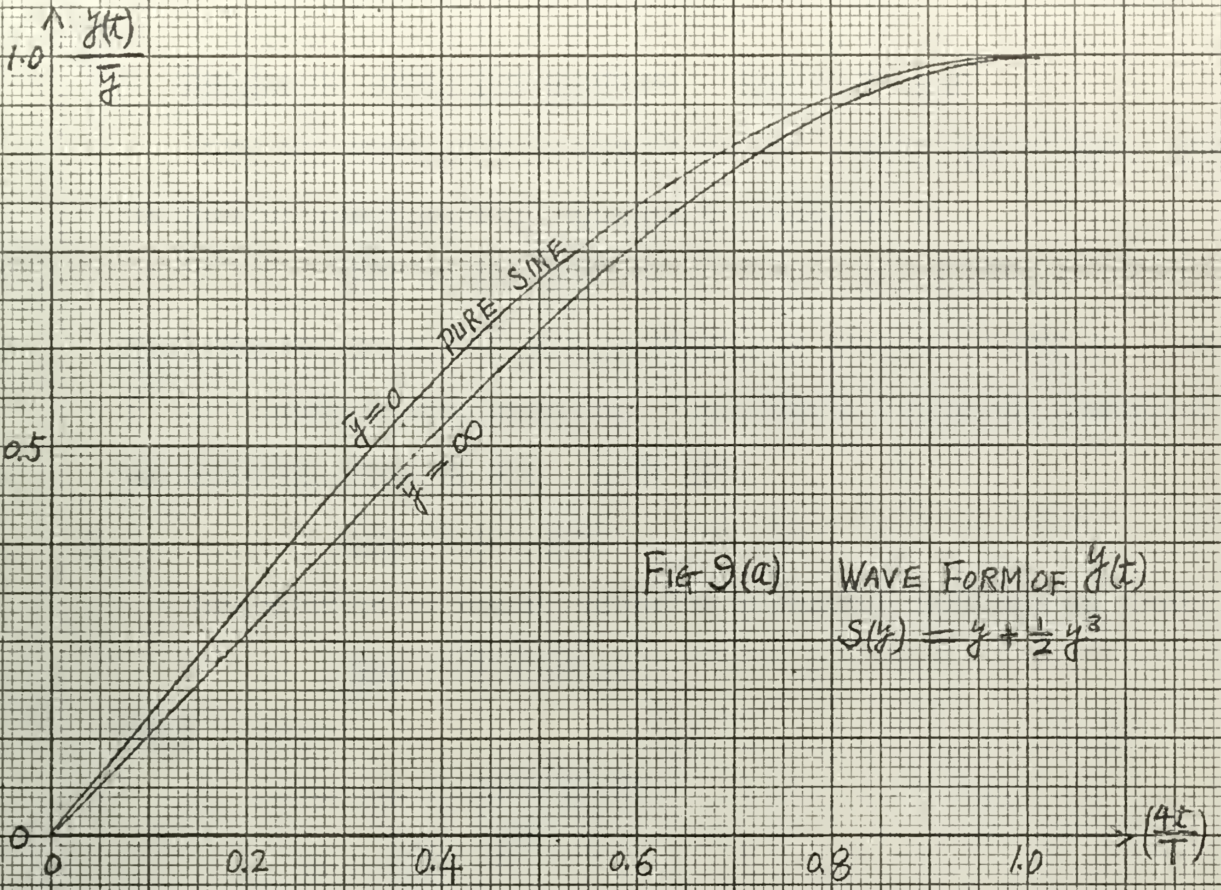


FIG 9(a) WAVE FORM OF $y(t)$

$S(y) = y + \frac{1}{2}y^3$

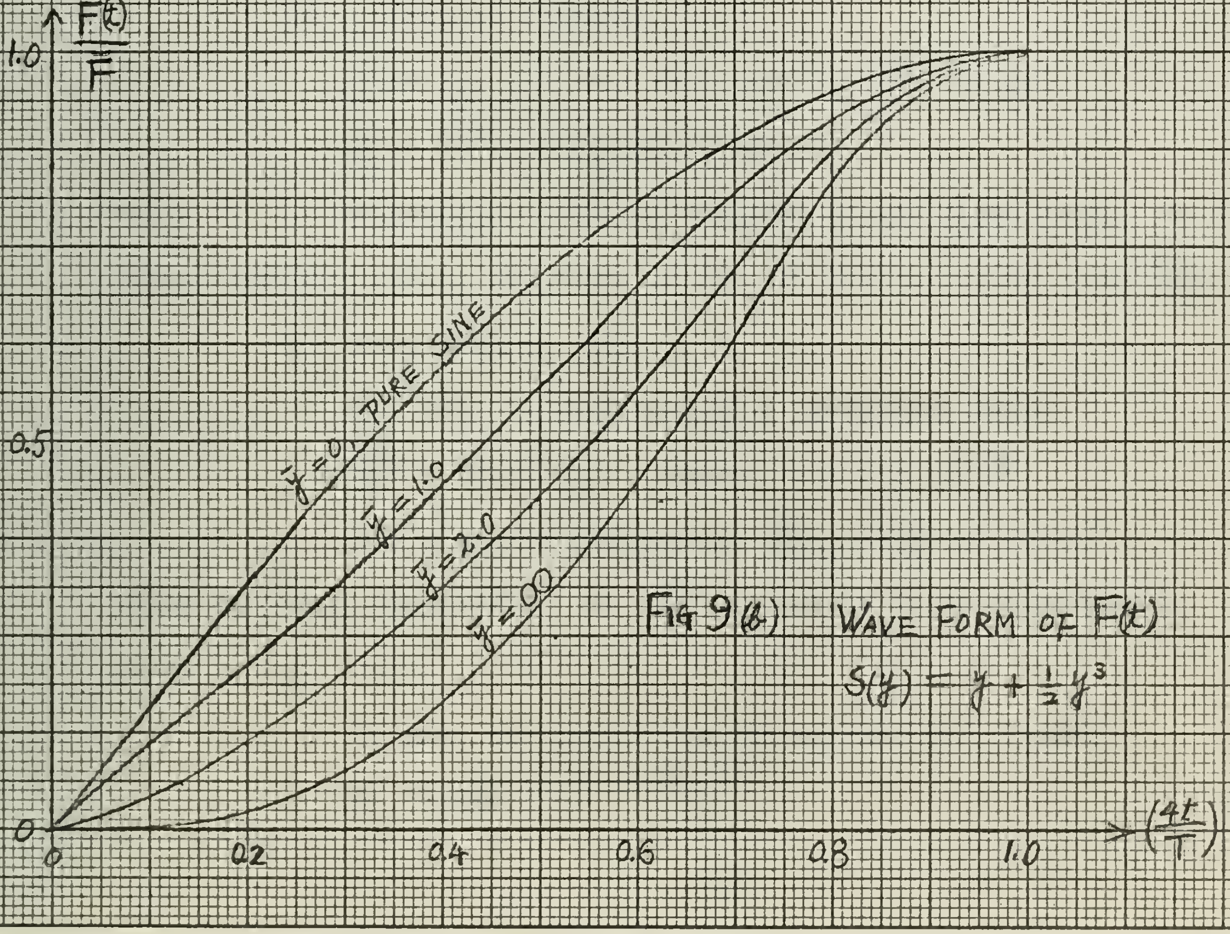


FIG 9(b) WAVE FORM OF $F(t)$

$S(y) = y + \frac{1}{2}y^3$

D. Example 4. Duffing's equation. $S(y) = y - \frac{1}{2}y^3$

When the spring characteristic is a softening type of cubic parabola, the general characteristic equation is expressed as

$$R(Y) = Y - \beta Y^3$$

which can be reduced the form of Table 1 by letting $Y = \frac{1}{\sqrt{2\beta}} y$, where y is a dimensionless quantity. Then without losing the generality of the governing equation (Eq. 3), we get

$$S(y) = y - \frac{1}{2}y^3$$

The solution of forced vibration is already tabulated in Table 1,

$$y(t) = \bar{y}\, sn(\omega t, \alpha) \tag{37}$$

where

$$\begin{aligned} \omega &= \omega_n\left(1 - \tfrac{1}{4}\bar{y}^2\right)^{1/2}\sqrt{1-\rho} \\ \alpha &= \bar{y}\left(4 - \bar{y}^2\right)^{-1/2} \\ \rho &= \frac{\bar{F}}{k\bar{S}}\sqrt{2\beta} \end{aligned} \tag{38}$$

(a) Frequency response.

From Eq. (37) and (38) with $T = \frac{4K}{\omega}$, we get

$$\frac{T}{T_n} = \left(\frac{2K}{\pi}\right)\left(\frac{1}{\sqrt{1-\frac{1}{4}\bar{y}^2}}\right)\left(\frac{1}{\sqrt{1-\rho}}\right) \tag{39}$$

where K is the complete elliptic integral of the first kind. The frequency response curves are shown in Fig. 10 for $\bar{F}$ = 0 and $\pm 0.2\, k/\sqrt{2\beta}$.

(b) Wave form of $y(t)$.

Eq. (37) is rewritten in a convenient form,

$$\frac{y(t)}{\bar{y}} = sn(\omega t, \alpha) \qquad \text{where} \quad \alpha = \bar{y}\left(4 - \bar{y}^2\right)^{-1/2}$$

For a small oscillation, $\lim_{\bar{y}\to 0} y(t)/\bar{y} = \sin \omega t$

which is purely sinusoidal. As $\bar{y}$ increases it is distorted as shown in Fig. 11(a).

To show the distortion the measure A is found,

$$A = \left(\frac{\pi}{2K}\right) \ln\left(1 + \frac{2\alpha}{1-\alpha}\right)^{\frac{1}{2\alpha}} \tag{40}$$

and is plotted in Fig. 10 along $\bar{y}$.

(c) Wave form of $F(t)$.

From Eq. (37) and the relation (5),

$$F(t) = \rho \frac{k}{\sqrt{2\beta}} S\left[y(t)\right] \quad \text{and} \quad \bar{F} = \rho \frac{k}{\sqrt{2\beta}} S(\bar{y})$$

rearranging these into a dimensionless form, we get

$$\frac{F(t)}{\bar{F}} = \frac{1}{1 - \frac{1}{2}\bar{y}^2}\left[\operatorname{sn}(\omega t, \alpha) - \frac{1}{2}\bar{y}^2 \operatorname{sn}^3(\omega t, \alpha)\right] \tag{41}$$

For a small oscillation, $\lim\limits_{\bar{y} \to 0} \frac{F(t)}{\bar{F}} = \sin \omega t$ which is sinusoidal. As $\bar{y}$ increases it is distorted as shown in Fig. 11(b). To compare this wave form with a sinusoidal one, the measure B is found,

$$B = \left(\frac{\pi}{2K}\right) \ln\left(1 + \frac{2\alpha}{1-\alpha}\right)^{\frac{1}{2\alpha}} + \left(\frac{\pi}{2K}\right)\left(\frac{4 - \bar{y}^2}{4 - 2\bar{y}^2}\right)\left[1 - \frac{1-\alpha^2}{\alpha} \cosh^{-1} \frac{1}{\sqrt{1-\alpha^2}}\right] \tag{42}$$

which is plotted along $\bar{y}$ in Fig. 10.

(d) Discussion.

This spring is of softening type, and at $\bar{y} = 0.82$ it reaches its maximum. After this point it drops down, and at $\bar{y} = \sqrt{2}$ the spring force becomes zero.

The measure B does not change appreciably up to $\bar{y} = 0.82$, and Fig. 11(b) shows that the wave form is not distorted much.

For any value of β which indicates the nonlinearity of the spring, the above solution can be applied.

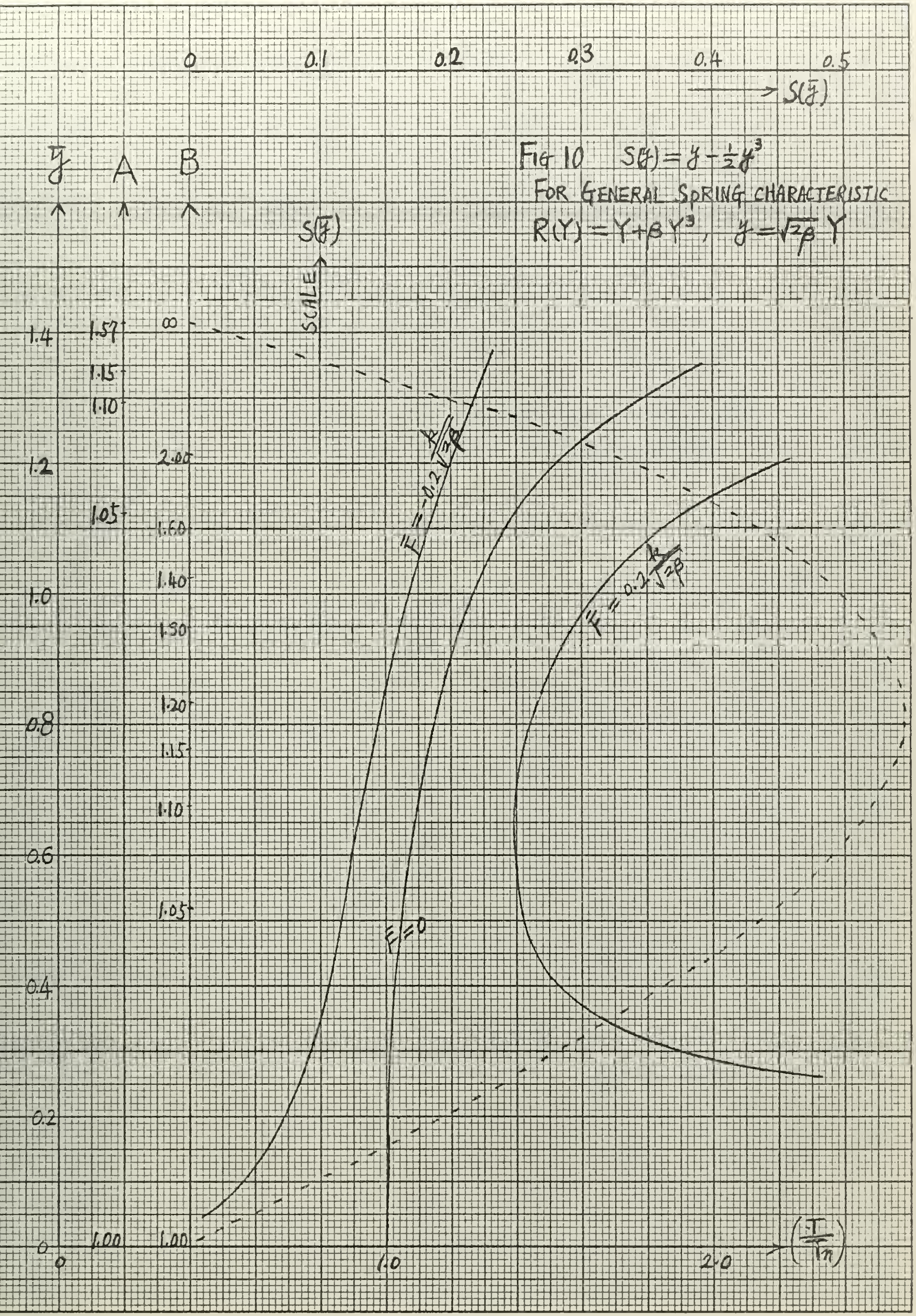


FIG 10 $S(\bar{y}) = y - \frac{1}{2}y^3$

FOR GENERAL SPRING CHARACTERISTIC

$R(Y) = Y + \beta Y^3, \quad y = \sqrt{2\beta}\, Y$

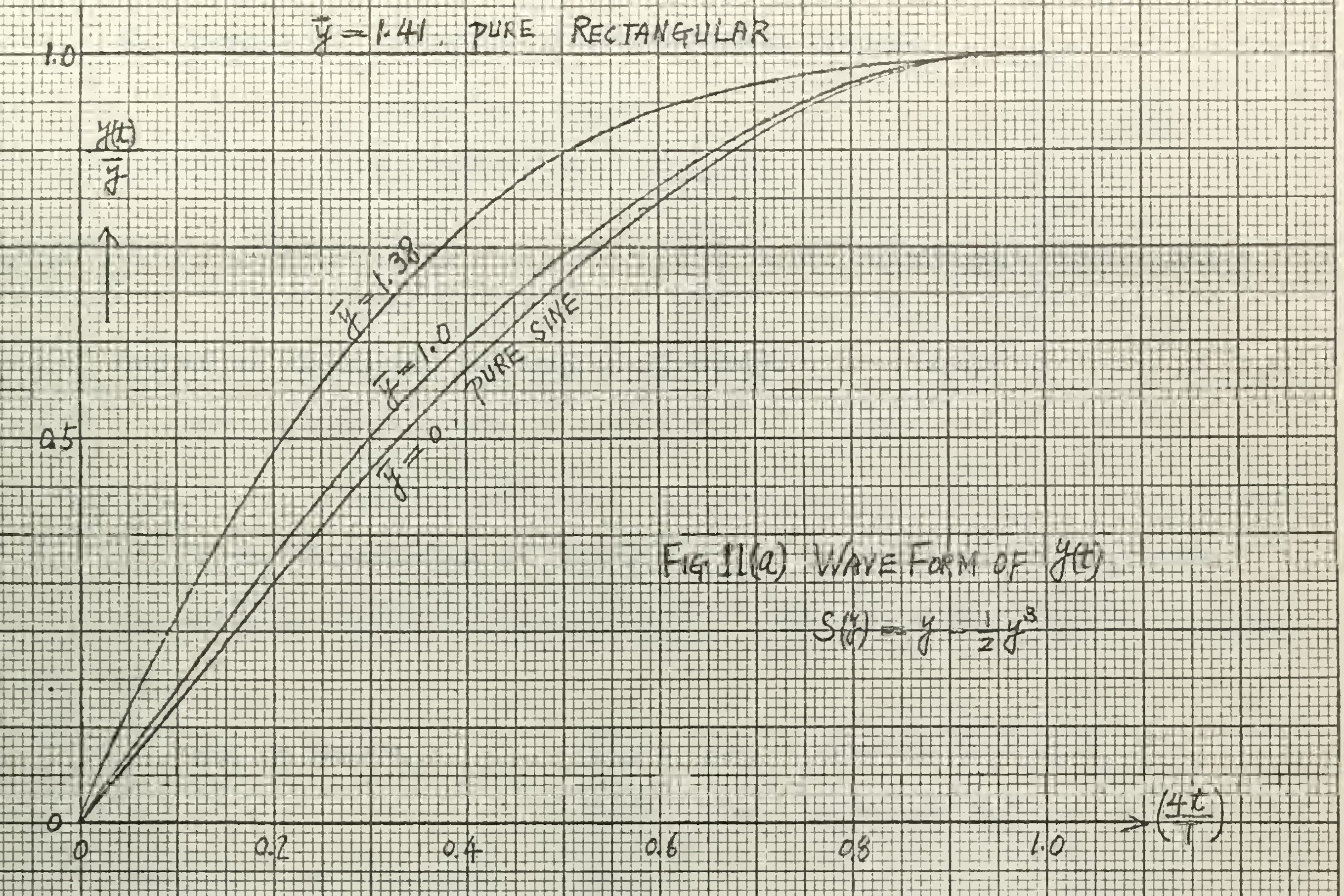


FIG 11(a) WAVE FORM OF $y(t)$

$S(y) = y - \frac{1}{2}y^3$

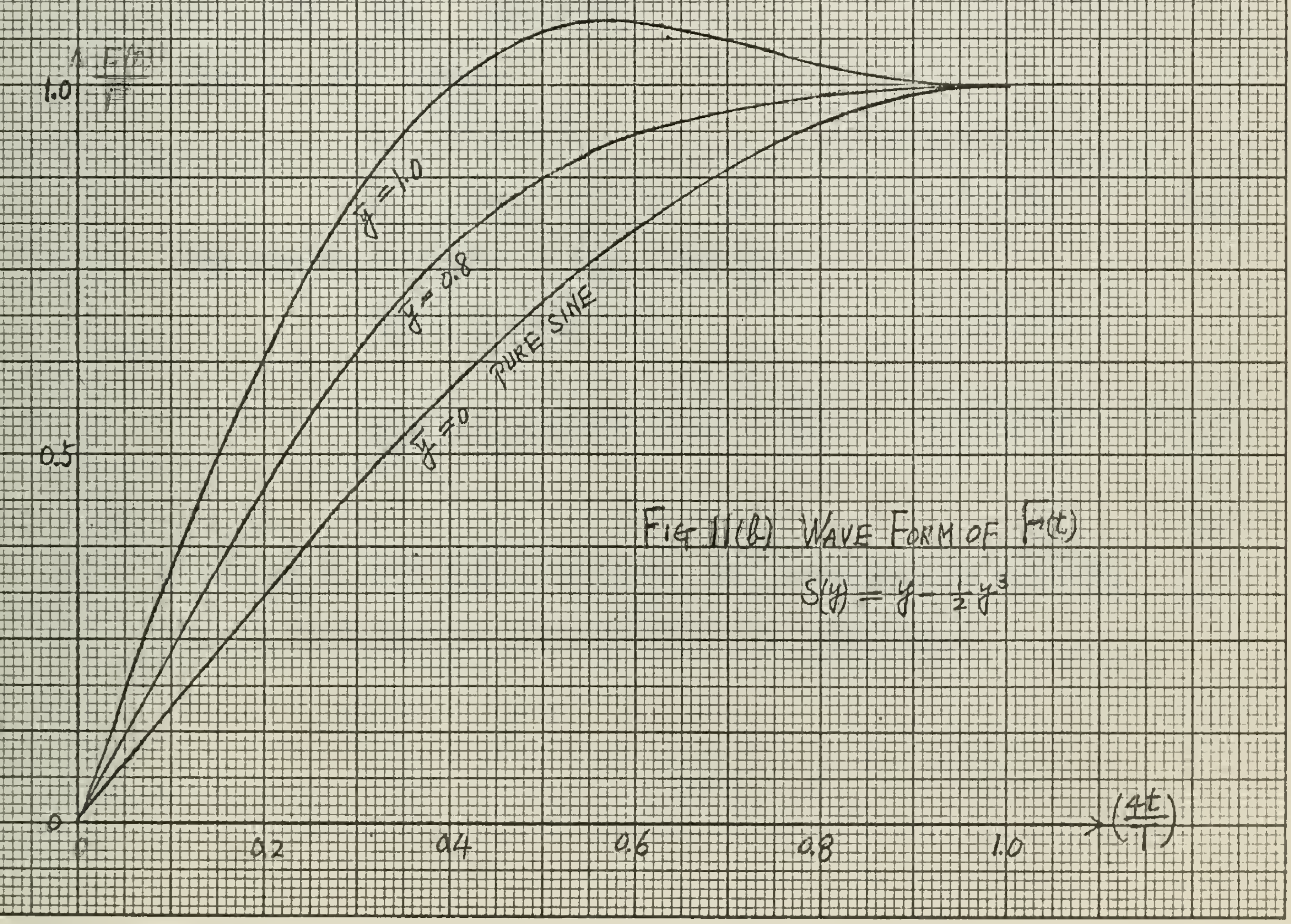
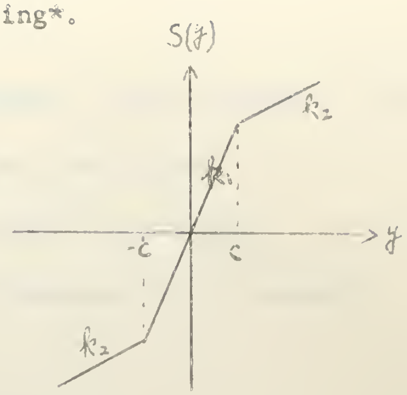


FIG 11(b) WAVE FORM OF $F(t)$

$S(y) = y - \frac{1}{2}y^3$

E. Example 5. Piecewise-linear softening spring*.



Suppose a symmetrical spring (as shown) is described by the equations,

$$S(y) = k_1 y \qquad |y| \leq c$$

$$S(y) = k_2 y + (k_1 - k_2) c \qquad |y| \geq c$$

and it is considered the case of $k_2 > 0$ only.

The equation of free vibration is

$$m\ddot{y} + S(y) = 0 \tag{43}$$

When the amplitude of deflection $\bar{y}$ is less than c, it is linear, and solution will not be included here. The case of $\bar{y} > c$ will be considered.

Let us denote $y_1(t)$ for $0 \leq t \leq t_c$

$y_2(t)$ for $t_c \leq t \leq T/4$

where t_c = the time when $y(t_c) = c$

and let $\omega_1^2 = k_1/m$ and $\omega_2^2 = k_2/m$

Then Eq. (43) becomes

$$\ddot{y}_1 + \omega_1^2 y_1 = 0 \qquad 0 \leq t \leq t_c$$

$$\ddot{y}_2 + \omega_2^2 y_2 + (\omega_1^2 - \omega_2^2) c = 0 \qquad t_c \leq t \leq T/4 \tag{43a}$$

The general solutions of Eqs. (43a) are:

$$y_1(t) = D_1 \sin(\omega_1 t + \phi_1) \qquad 0 \leq t \leq t_c$$

$$y_2(t) = D_2 \sin(\omega_2 t + \phi_2) + (1 - \frac{\omega_1^2}{\omega_2^2}) \qquad t_c \leq t \leq T/4 \tag{44}$$

---

* In section E and F (which follows), for convenience in notation, only the first quarter cycle of motion is considered.

where D_1, D_2, ϕ_1, and ϕ_2, are arbitrary constants, which together with t_c may be determined by the following boundary conditions;

$y(0) = 0$ as an initial condition.

$\dot{y}^2(0) = 2U(\bar{y})$ by the conservation of energy.

$y_1(t_c) = y_2(t_c) = c$

$\dot{y}_1(t_c) = \dot{y}_2(t_c)$

Solving for the constants in Eqs. (44), we get

$$D_1 = \left[c^2 + 2c(\bar{y}-c) + \frac{\omega_2^2}{\omega_1^2}(\bar{y}-c)^2\right]^{1/2}$$

$$D_2 = -c\left(1 - \frac{\omega_1^2}{\omega_2^2}\right) + \bar{y} \tag{45}$$

$$t_c = \frac{1}{\omega_1}\sin^{-1}\left(\frac{c}{D_1}\right)$$

$$\phi_2 = \sin^{-1}\left(\frac{\omega_1^2 c}{\omega_2^2 D_2}\right) - \omega_2 t_c \quad , \quad \phi_1 = 0$$

To obtain the frequency response curves, the period of free oscillation ($\rho = 0$) T_0 can be found:

At $t = T_0/4$, $\dot{y}_2(T_0/4) = 0$

because $y_2(t)$ reaches its maximum value of $\bar{y}$ at $t = T_0/4$. Solving for T_0 we get:

$$T_0 = \frac{4}{\omega_2}\left(\frac{\pi}{2} - \phi_2\right)$$

$$\frac{T_0}{T_n} = \left(\frac{\omega_1}{\omega_2}\right)\left(1 - \frac{\phi_2}{\pi/2}\right)$$

where $T_n = \frac{2\pi}{\omega_1}$ is the period of a small free oscillation.

By making use of Eq. (16) for forced vibrations, the period of forced vibration is obtained,

$$\frac{T}{T_n} = \left(\frac{\omega_1}{\omega_2}\right)\left(1 - \frac{\phi_2}{\pi/2}\right)\frac{1}{\sqrt{1-\rho}} \tag{46}$$

As a specific example let us take $k_1 = 4k_2$ in the above symmetrical spring. Then we get $\omega_1 = 2\omega_2$, and the solutions of Eqs. (44) become

$$y_1(t) = D_1 \sin \omega_1 t \qquad 0 \le t \le t_c$$

$$y_2(t) = (\bar{y} + 3c)\sin(\omega_2 t + \phi_2) - 3c \qquad t_c \le t \le T/4 \tag{44a}$$

where

$$D_1 = \left[c^2 + 2c(\bar{y}-c) + \frac{1}{4}(\bar{y}-c)^2\right]^{1/2}$$

$$t_c = \frac{1}{\omega_1} \sin^{-1}\left(\frac{c}{D_1}\right)$$

$$\phi_2 = \sin^{-1}\left(\frac{4c}{\bar{y}+3c}\right) - \frac{1}{2}\sin^{-1}\left(\frac{c}{D_1}\right)$$

(a) Frequency response.

From Eq. (46),

$$\frac{T}{T_n} = \left(2 - \frac{4\phi_2}{\pi}\right) \frac{1}{\sqrt{1-\rho}} \tag{46a}$$

where ρ is an assignable constant as defined in the relation (5). In Fig. 12 the frequency response curves are shown for $\bar{F}=0$, $\pm 0.5 k_1 c$, and $\pm k_1 c$.

(b) Wave form of $y(t)$.

Eq. (44a) is rewritten.

$$\begin{aligned} \frac{y(t)}{\bar{y}} &= \frac{D_1}{\bar{y}} \sin \omega_1 t & 0 \leq t \leq t_c \\ y(t)/\bar{y} &= \left(1+\frac{3c}{\bar{y}}\right) \sin(\omega_2 t + \phi_2) - \frac{3c}{\bar{y}} & t_c \leq t \leq T/4 \end{aligned} \tag{44b}$$

For $\bar{y} \leq c$ the motion is purely sinusoidal, and as $\bar{y}$ increases it is distorted. At around $\bar{y} = 2.5c$, distortion reaches a maximum, and for further increase in $\bar{y}$ the wave form approaches a sinusoidal form again. In Fig. 13(a) these two wave forms are shown: one is sinusoidal and the other shows the maximum distortion at $\bar{y} = 2.5c$.

To show the distortion the measure A is found,

$$\begin{aligned} A = {} & \frac{D_1}{\bar{y}}(1-\cos\omega_1 t_c)\frac{\pi/4}{\pi/2-\phi_2} + \left(1+\frac{3c}{\bar{y}}\right)\cos(\omega_2 t_c+\phi_2)\frac{\pi/2}{\pi/4-\phi_2} \\ & - \frac{\pi}{2}\left(\frac{3c}{\bar{y}}\right)\left(1-\frac{\omega_2 t_c}{\pi/2-\phi_2}\right) \end{aligned} \tag{47}$$

which is plotted in Fig. 12 along $\bar{y}/c$. Actually the distortion is so small that the measure A is not much greater than unity, where the maximum measure A = 1.023 at $\bar{y} = 2.5c$.

(c) Wave form of $F(t)$.

From Eq. (45) and the relation (5), and with the spring force $S(y) = (3c + y)k_2$, $|y| \geq c$, for this spring, we get

$$\frac{F(t)}{\bar{F}} = \frac{4D_1}{\bar{y}+3} \sin \omega_1 t_c \qquad 0 \leq t \leq t_c$$

$$\frac{F(t)}{\bar{F}} = \sin(\omega_2 t + \phi_2) \qquad t_c \leq t \leq T/4 \tag{48}$$

In Fig. 13(b) the wave form is plotted for $\bar{y} \leq c$, $\bar{y} = 2c$ and $5c$. Since the spring force has a slope discontinuity at $y = c$, the wave form of $F(t)$ is slope discontinuous at $t = t_c$.

For $\bar{y} \leq c$ it is a pure sinusoidal form. As $\bar{y}$ increases it becomes distorted; maximum distortion occurs at $\bar{y} = 2.5c$ approximately. Further increases in $\bar{y}$ bring out a sinusoidal form again, similar to $y(t)$ wave forms. Every wave form of $F(t)$ does not go beyond a limiting curve which is marked as "Envelope" in Fig. 13(b).

To compare the wave form of $F(t)$ with a sinusoidal form, the measure B is found,

$$B = \frac{4D}{\bar{y}+3c}(1-\cos\omega_1 t_c)\frac{\pi/4}{\pi/2-\phi_2} + \frac{\pi/2}{\pi/2-\phi_2}\cos(\omega_2 t_c + \phi_2) \tag{49}$$

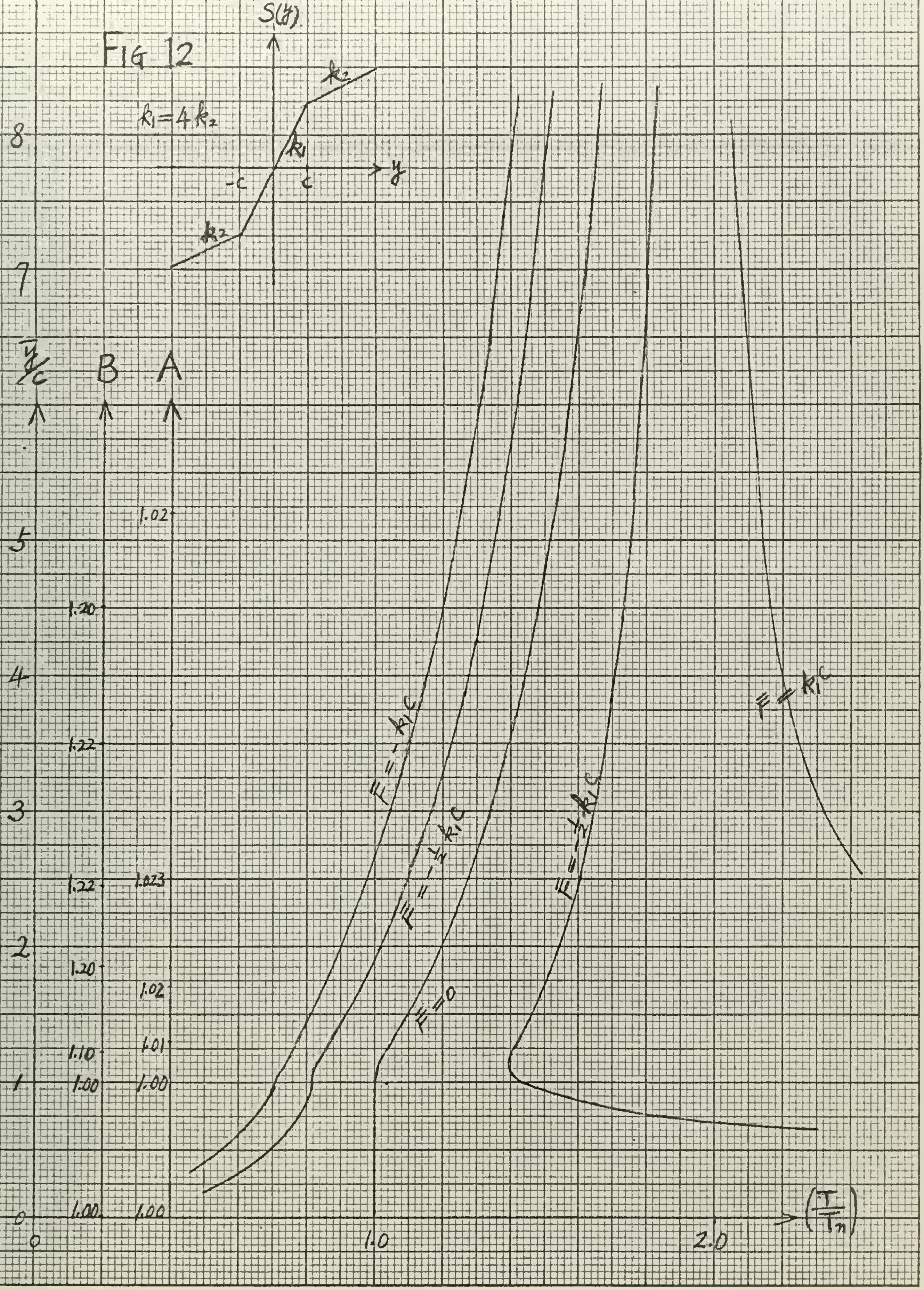
which is plotted along $\bar{y}/c$ in Fig. 12. At $\bar{y} = 2.5c$, approximately, B reaches its maximum value of 1.22. At this point the distortion is a maximum.

(d) Discussion.

The solution of free vibration is found by a classical method, and for the forced vibration Harvey's method is applied to that of free vibration.

Fig. 13(b) shows that $F(t)$ is not far away from a pure sinusoidal form, and the measure B is not much greater than unity.

FIG 12


S(y)
k2
k1 = 4 k2
k1
-c
c
y
k2
8
7
y/c
B
A
1.02
5
1.20
4
1.22
3
1.22
1.023
2
1.20
1.02
1.10
1.01
1
1.00
1.00
0
1.00
1.00
0
1.0
2.0
(T/Tn)
F = -k1c
F = -1/2 k1c
F = 0
F = 1/2 k1c
F = k1c

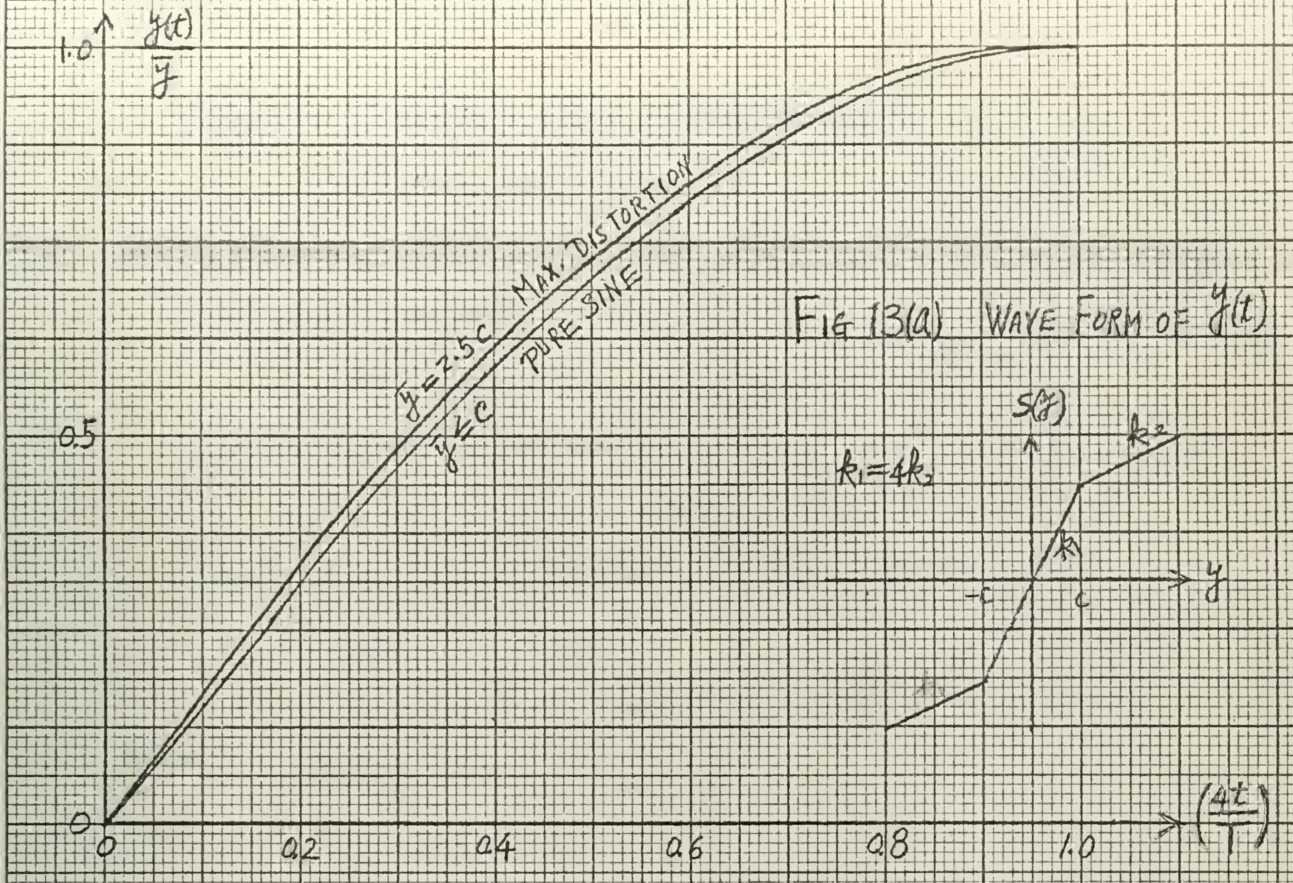


FIG 13(a) WAVE FORM OF $y(t)$

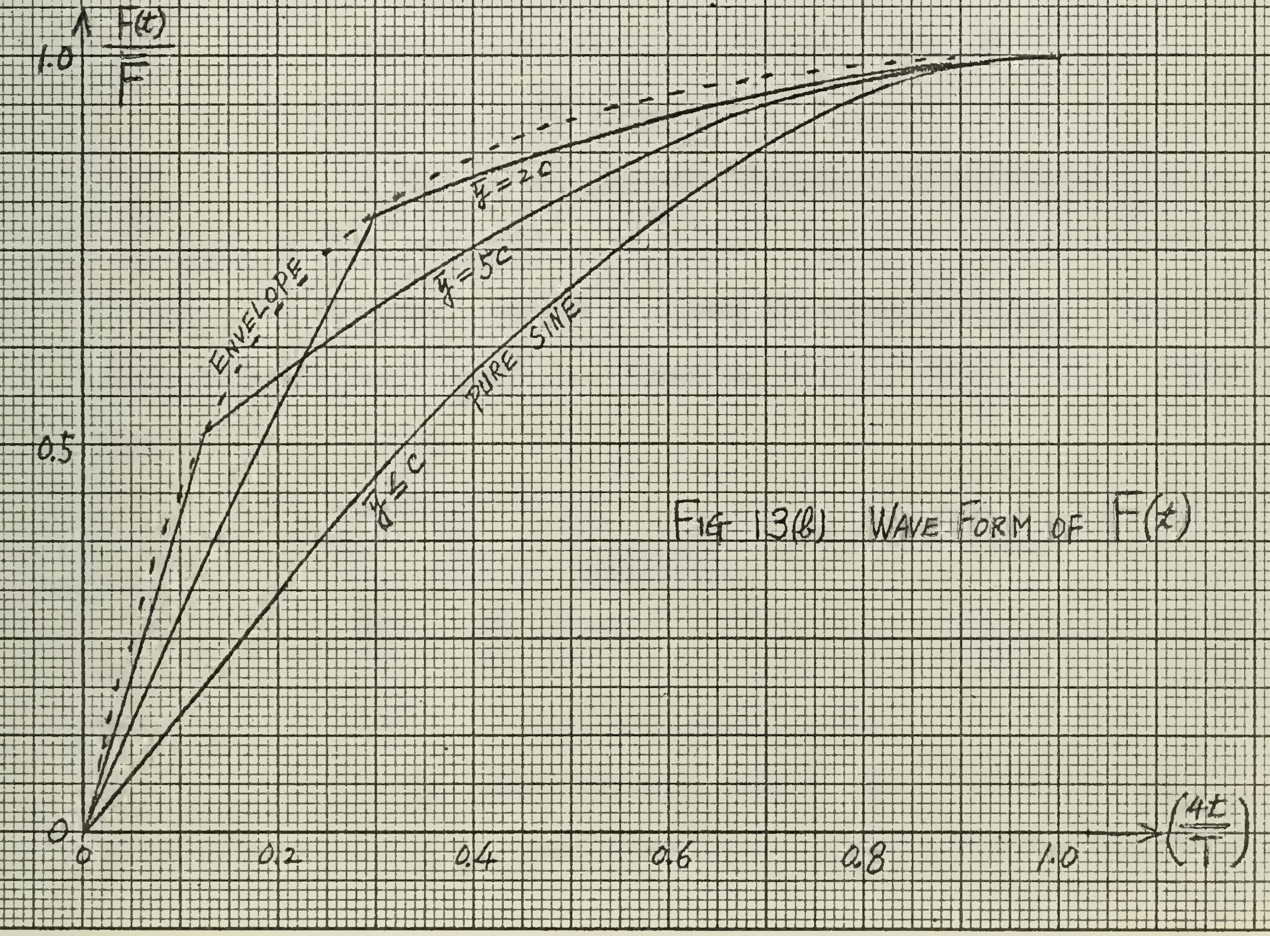
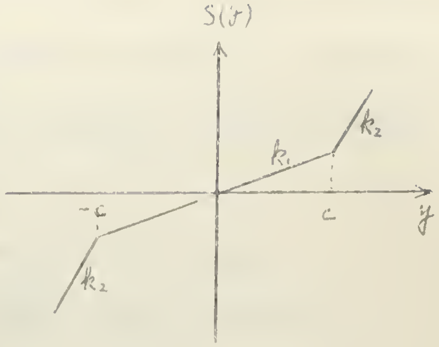


FIG 13(b) WAVE FORM OF $F(t)$

F. Example 6. Piecewise linear hardening spring.

Let us consider a symmetrical spring as shown. Let $4k_1 = k_2$ then $2\omega_1 = \omega_2$



For free vibrations,

$$m\ddot{y} + S(y) = 0$$

Let the initial conditions be

$$y(0) = 0 \quad , \quad \dot{y}^2(0) = 2U(\bar{y})$$

Then the solution is obtained from Eqs. (44) and (45),

$$y(t) = D_1 \sin \omega_1 t \qquad 0 \leq t \leq t_c$$

$$y(t) = (\bar{y} - \tfrac{3}{4}c) \sin(\omega_2 t + \phi_2) + \tfrac{3}{4}c \qquad t_c \leq t \leq T/4 \tag{50}$$

where

$$D_1 = \left[c^2 + 2c(\bar{y} - c) + 4(\bar{y} - c)\right]^{1/2}$$

$$t_c = \frac{1}{\omega_1} \sin^{-1}\left(\frac{c}{D_1}\right)$$

$$\phi_2 = \sin^{-1}\left(\frac{c}{4\bar{y} + 3c}\right) - 2\sin^{-1}\left(\frac{c}{D_1}\right)$$

(a) Frequency response.

From Eq. (46) and with $2\omega_1 = \omega_2$, we get

$$\frac{T}{T_n} = \left(\frac{1}{2} - \frac{\phi_2}{\pi}\right)\frac{1}{\sqrt{1-\rho}} \tag{51}$$

In Fig. 14 the frequency response curves are plotted for $\bar{F}$ = 0, ± $k_1 c$, and ±$5k_1 c$. For $\bar{y} \leq c$, it is linear and the ratio (T_0/T_n) is unity. And as $\bar{y}$ becomes much greater than C , the spring characteristic again approaches linearity and the ratio (T_0/T_n) approaches 0.5.

(b) Wave form of $y(t)$.

Eqs. (50) are rewritten:

$$\frac{y(t)}{\bar{y}} = \frac{D_1}{\bar{y}} \sin \omega_1 t \qquad 0 \leq t \leq t_c$$

$$\frac{y(t)}{\bar{y}} = \left(1 - \frac{3c}{4\bar{y}}\right) \sin(\omega_2 t_c + \phi_2) + \frac{3c}{4\bar{y}} \qquad t_c \leq t \leq T/4 \tag{52}$$

In Fig. 15(a) the wave form of $y(t)$ is shown for $\bar{y} \leq c$ and $\bar{y} = 15c$. For $\bar{y} \leq c$, the spring is linear and the wave form is purely

sinusoidal. When $\bar{y}$ becomes greater than C, the wave form is distorted, at approximately $\bar{y} = 1.5C$ the distortion reaches a maximum. For further increase in $\bar{y}$, the spring characteristic approaches linearity again, the wave form approaches a sinusoidal form.

To show the distortion the measure A is found,

$$A = \frac{\pi}{\pi/2-\phi_2}\left(\frac{D_1}{\bar{y}}\right)\left(1-\cos\omega_1 t_c\right) + \frac{\pi/2}{\pi/2-\phi_2}\left(1+\frac{3c}{4\bar{y}}\right)\cos(\omega_2 t_c+\phi_2)$$
$$+ \frac{\pi}{2}\left(\frac{3c}{4\bar{y}}\right)\left(1-\frac{\omega_2 t_c}{\pi/2-\phi_2}\right) \qquad (53)$$

which is plotted along $\bar{y}/c$ in Fig. 14. At $\bar{y}=1.5C$ the measure A reaches a minimum (A = 0.965).

(c) Wave form of $F(t)$.

The spring characteristic is

$$S(y) = k_2 y - \tfrac{3}{4}k_2 c \quad \text{for} \quad |y| \geq c$$

Substituting Eqs. (50) into the above equation, we get

$$\frac{F(t)}{\bar{F}} = \frac{D_1}{4\bar{y}-3c}\sin\omega_1 t \qquad 0 \leq t \leq t_c$$
$$\frac{F(t)}{\bar{F}} = \sin(\omega_2 t+\phi_2) \qquad t_c \leq t \leq T/4 \qquad (54)$$

In Fig. 15(b) the wave form of $F(t)$ is shown for $\bar{y} \leq C$, $\bar{y} = 1.5C$, and 3.0 C. There is a limit, beyond which the wave form does not go, as indicated as "Envelope" in the figure.

To compare the $F(t)$ wave form with a sinusoidal form the measure B is found,

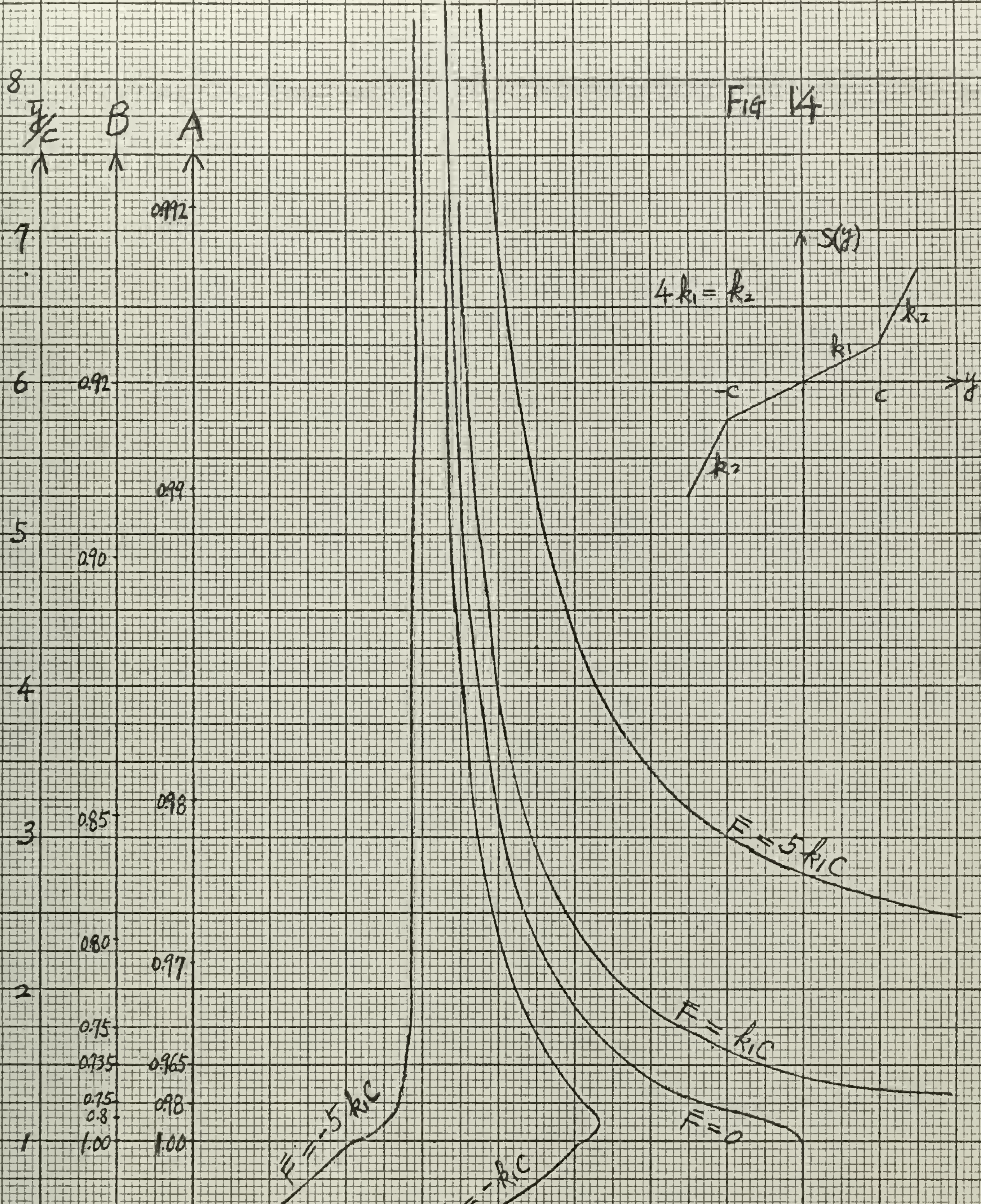
$$B = \frac{D_1}{4\bar{y}-3c}\left(1-\cos\omega_1 t_c\right)\frac{\pi}{\pi/2-\phi_2} + \frac{\pi/2}{\pi/2-\phi_2}\cos(\omega_2 t_c+\phi_2) \qquad (55)$$

which is plotted in Fig. 14 along $\bar{y}/c$. For $\bar{y} \leq c$ the spring is linear and $F(t)$ is a pure sinusoidal form, so the measure B is unity. As $\bar{y}$ goes beyond C, the measure B decreases rapidly, and at $\bar{y}=1.5c$, approximately, B = 0.735 as a minimum. For further increase in $\bar{y}$, the measure B slowly approaches unity where the spring characteristic becomes linear again.

(d) Discussion.

This example is similar to the previous one, from which the general solution is obtained. The measure B and the wave form of $F(t)$ shown in Fig. 15(b) tells us that the external force ("natural") is not so far away from a sinusoidal form.

Fig 14



8
7
6
5
4
3
2
1
0
T_y/C
B
A
0.992
0.92
0.99
0.90
0.85
0.98
0.80
0.97
0.75
0.735
0.965
0.75
0.8
0.98
1.00
1.00
1.0
1.0
0
0.4
0.6
0.8
1.0
$\left(\frac{T}{T_n}\right)$
$F = 5k_1C$
$F = k_1C$
$F = 0$
$F = -k_1C$
$F = -5k_1C$
$S(y)$
$4k_1 = k_2$
k_2
k_1
k_2
$-c$
c
y

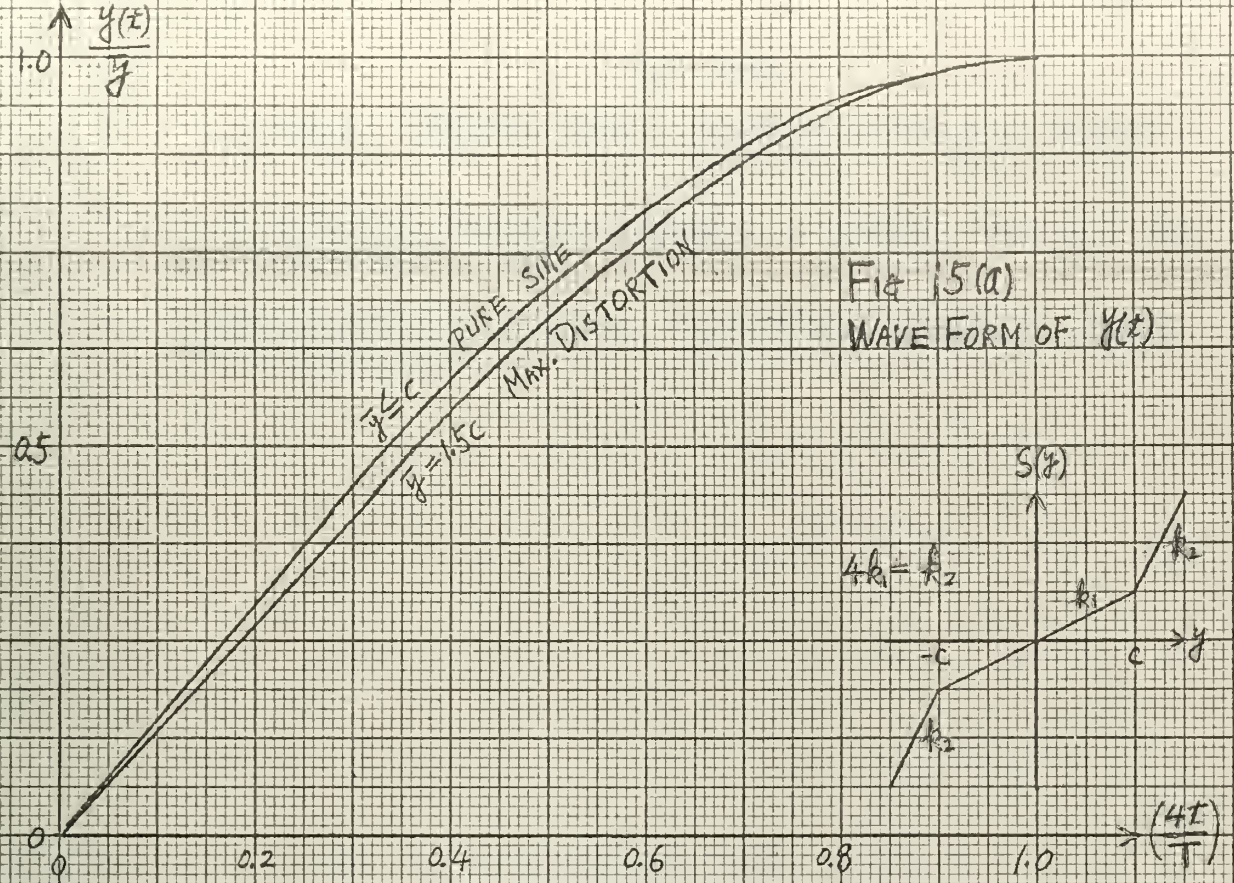


FIG 15(a)
WAVE FORM OF $y(t)$

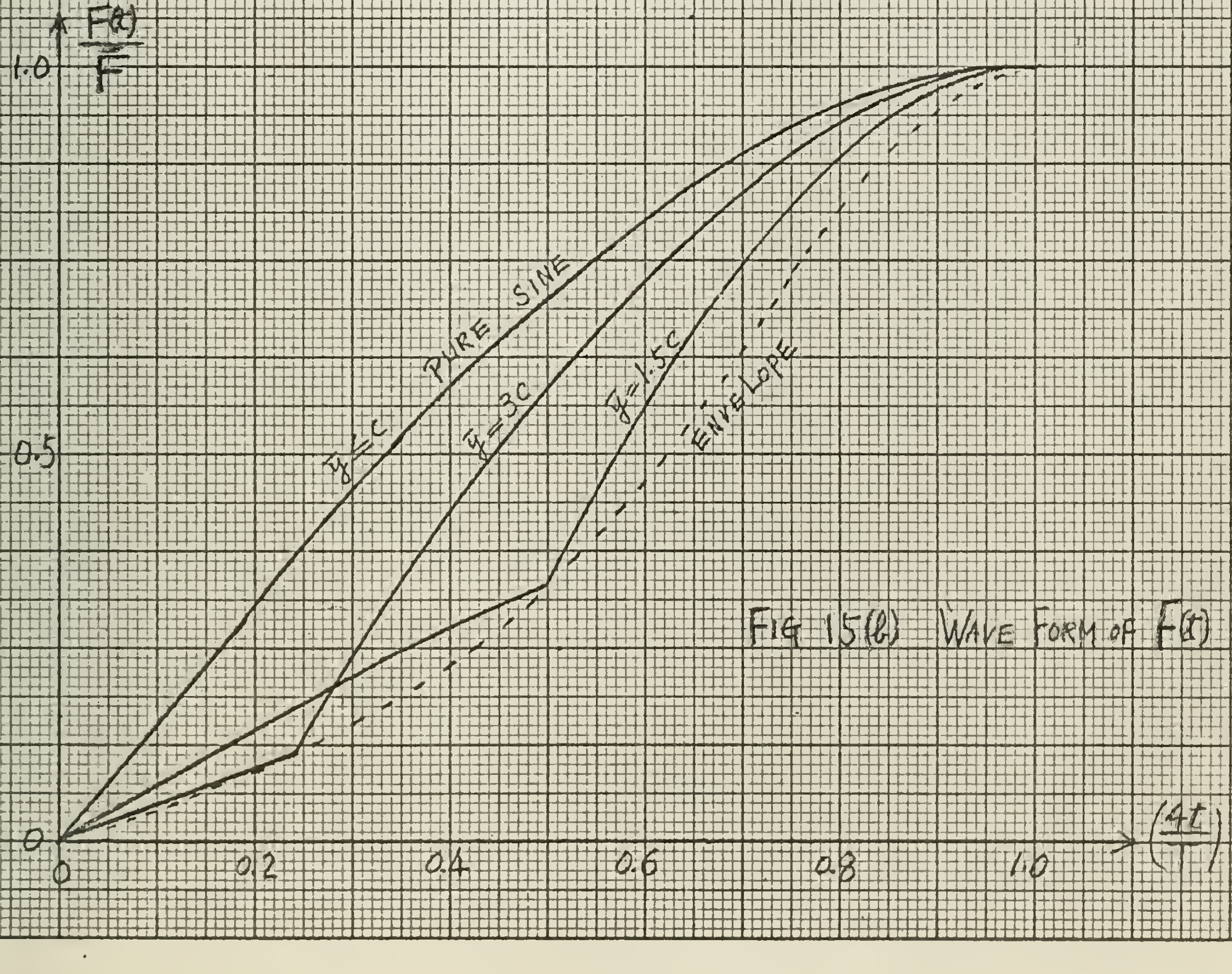


FIG 15(b) WAVE FORM OF $F(t)$

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